- Neutons law of cooling:- $\boxed{\frac{d\theta}{dt} = -k(\theta - \theta_s)}$

θ – temp at t
θ_s – room temperature or final temperature.
$-k$ Neutons constant, proportionality constant.

Solution
$$\theta = c_1 e^{-kt} + \theta_s$$

- $f(D)y = a^x$

to find PI $\rightarrow$ $\boxed{a^x = e^{x(\log a)}}$

CF: $f(D)y = 0$

$$PI = \frac{1}{f(D)} e^{x(\log a)} = \frac{1}{f(\log a)} e^{x(\log a)}$$

- Even though the entire equation may still be non linear put xy or $(x+y)$ etc $= u$ in trignometric functions

When checking for linear
- $\rightarrow$ Check linear in y
- $\rightarrow$ Check if non linear - bernoulies
- $\rightarrow$ check linear in x
- $\rightarrow$ Check if non linear - bernoulie

First degree quadratic equations check:
- $\rightarrow$ Variable seperable first
- $\rightarrow$ Homogeneous
- $\rightarrow$ Linear
- $\rightarrow$ Exact conventional
- $\rightarrow$ Non Exact.

Differential Equations (D.E)

Order : The order of highest derivative involved in the differential eq is called order of the differential eq.

Degree : The highest power of highest derivative free from radicals or fractional powers is called degree of differential eq.
provided the derivatives of dependent variable should be free from radicals & fractional ~~powers~~

eg.

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[x - \left(\frac{dy}{dx}\right)^6\right]^{1/2}$$

$$\frac{d^2y}{dx^2}^4 = x - \left(\frac{dy}{dx}\right)^6$$

order = 2
degree = 4

Fractions

$$\frac{x}{\frac{dy}{dx}}, \quad \frac{x^2}{\frac{dy}{dx}}, \quad \frac{1}{\frac{d^3y}{dx}}$$

Radical

$$\left(\frac{dy}{dx}\right)^{1/n}$$

Formation of D.E

$F(x,y,a,b) = 0$ —— (1)

diff wrt $x \longrightarrow F'\left(x,y,\frac{dy}{dx},a,b\right) = 0$ —— (2)

diff wrt $x \longrightarrow F''\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},a,b\right) = 0$ —— (3)

Eliminate a,b from (3) ((1), (2), (3) using)
We get D.E
generally
differentiate n times
where n is the no. of arbitrary constants.

→ The no. of arbitrary constants eliminated should be equal to the order of the resulting ordinary differential equation.

---

eg:- Find the di. eq of family of curves $y = \frac{a}{x^2} + bx$ (a, b - arbitrary const)

$$\Rightarrow y' = \frac{-2a}{x^3} + b \Rightarrow y' = \frac{-2}{x}(y - bx) + b \Rightarrow y' = \frac{-2y}{x} + 3b$$

$$\Rightarrow y'' = \frac{x(-2y') + 2y}{x^2} \Rightarrow x^2y'' = -2xy' + 2y \Rightarrow \underline{x^2y'' + 2xy' - 2y = 0}$$

General Solution ⟶ Particular Solution
⟶ Singular Solution.

$y^2 = 4ax$: parabola

$(x-a)^2 + (y-b)^2 = c^2$: circle

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$: hyperbola

$\boxed{\text{subset } xy = c}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipse

First Order First Degree D.E

$$\boxed{\frac{dy}{dx} = F(x,y)} \quad \text{or} \quad \boxed{M(x,y)\,dx + N(x,y)\,dy = 0}$$

→ We use the following methods to find the solutions of first order first degree D.E.

1. Variable-Separable form ★
2. Homogeneous D.E
3. Exact differential equation
4. Non Exact D.E
5. Linear D.E ★
6. Non-Linear D.E

Variable Separable form

$$\boxed{\frac{dy}{dx} = \frac{f(x)}{g(y)}}$$

$$g(y)\,dy = f(x)\,dx$$

$$\boxed{\int g(y)\,dy = \int f(x)\,dx + C}$$

eg.) 4) $\frac{dy}{dx} + 7x^2 y = 0$ with $y(0) = 3/7$ find $y(1) = $ ______

Ans) $\frac{dy}{dx} = -7x^2 y \Rightarrow \int \frac{1}{y}\,dy = -7\int x^2\,dx$

$\Rightarrow \log y = \frac{-7x^3}{3} + \log c$

$y = ce^{-\frac{7}{3}x^3}$

$y(0) = c = 3/7$

$\therefore y = \frac{3}{7} e^{-7/3\, x^3}$

$y(1) = \frac{3}{7} e^{-7/3}$

Note: constant of integration can be $\log c$, e^c, etc for further simplification

eg) $xy\frac{dy}{dx} = x + y + xy + 1$

might seem to be inseparable but sometimes simple reduction can be done.

$xy\frac{dy}{dx} = x + 1 + y(1+x) = (x+1)(y+1)$

factorising can simplify to variable separable.

$\rightarrow \int \frac{y}{1+y}\,dy = \int \frac{x+1}{x}\,dx$

$\int \left(1 - \frac{1}{1+y}\right) dy = \int \left(1 + \frac{1}{x}\right) dx \Rightarrow y - \log(1+y) = x + \log x + C$

eg) The solution of $\frac{dy}{dx} = -\left(\frac{x}{y}\right)^n$ for $n=1$ & $n=-1$ represent

$\underline{n=-1}$

$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow +\log x + \log y = \log c$

$\Rightarrow \log xy = \log c \Rightarrow \underline{xy = c}$

(hyperbola)

$\underline{n=1}$

$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y^2 = -x^2 + c$

$x^2 + y^2 = c$ (circle)

standard family of curves must be known

eg.1) The general sol of $y\,dx + (1+x^2)(1+\log y)\,dy = 0$

Sol)

$y\,dx = -(1+x^2)(1+\log y)\,dy$

$\frac{dx}{(1+x^2)} = -\left(\frac{1+\log y}{y}\right)dy \Rightarrow \tan^{-1}x = -\log y$

$-\int t\,dt$

$\log y = t$

$\frac{dt}{dy} = \frac{1}{y}$

$dt = \frac{1}{y}dy$

$\Rightarrow \tan^{-1}x + \log y + \frac{t^2}{2} + c = 0$

$\Rightarrow \tan^{-1}x + \log y + \frac{(\log y)^2}{2} = c$

eg 3) Find the curve passing through the origin and satisfying $\frac{dy}{dx} + y = 1$

$\frac{dy}{dx} = 1-y \Rightarrow \frac{dy}{(1-y)} = dx \Rightarrow x = -\log(1-y) + \log c$

$\Rightarrow e^x = \cancel{e^{c(1-y)^{-1}}}\ \frac{c}{1-y} \Rightarrow$

$\underline{y = (1-e^{-x})}$

$(1-y)e^x = c \Rightarrow e^x - e^x y = c$

$y = \frac{-c + e^x}{+e^x} \Rightarrow \frac{d}{e^x} \cancel{+1}$

$y = -ce^{-x} + 1 = (1 - ce^{-x})$

$y(0) = 0 = 1 - c \times 1 \Rightarrow c = 1$

$\therefore \underline{y = 1 - e^{-x}}$

→ If the given equation contains terms like $\cos(xy)$, $\sec(x+y)$, $(ax+by+c)^2$ Can be reduced to variable separable form with substitutions $xy=V$, $x+y=V$, $ax+by+c=V$ respectively

eg:2) $\frac{dy}{dx}=(x+y-1)^2$

Sol. put $(x+y-1)=u \Rightarrow \frac{du}{dx}=1+\frac{dy}{dx}$

$\frac{du}{dx}-1=(x+y-1)^2 = \frac{du}{dx}-1=u^2$

$\Rightarrow du=(u^2+1)\,dx \Rightarrow \frac{du}{u^2+1}=dx$

~~$u=\frac{u^3}{3}+u$~~

$\tan^{-1}u = x+c$

$\tan^{-1}(x+y-1)=x+c$

$\Rightarrow x+y-1=(\tan x+c)$

$y=\tan(x+c)-x+1$

Homogeneous D.Eq

→ The D.Eq $\frac{dy}{dx}=f(x,y)$ is said to be a homogeneous differential equation if $f(x,y)$ is a homogeneous function of degree zero.

→ The differential eq. $Mdx+Ndy=0$ is said to be a homogeneous differential equation, if all the terms of M & N are of same degree.

→ Every homogeneous function of degree zero can be written as a function of y/x or x/y and the substitution (ie $\frac{y}{x}=V$ or $\frac{x}{y}=V$) reduces the given equation to variable separable form.

eg 5:- $\frac{dy}{dx} = \frac{3x^2-2xy}{x^2}$ sol ~~put~~

put $y = vx$

$$\frac{dy}{dx} = \frac{3x^2-2vx^2}{x^2} = 3-2v$$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v = 3-2v$$

$$x\frac{dv}{dx} = 3-2v-v$$

$$\Rightarrow \frac{dv}{dx} = \frac{3-3v}{x}$$

$$\int\frac{dv}{1-v} = \int\frac{3}{x}dx$$

$$3\log x + \log c = -\log(1-v)$$

$$cx^3 = \frac{1}{(1-v)} \Rightarrow \left(1-\frac{y}{x}\right) = \frac{c}{x^3}$$

$$x^3 - x^2y = c$$

$$(y-x)x^2 = c$$

6/ $\frac{dy}{dx} = \left(\frac{y}{x}\right) + \sec(y/x)$

Note:) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ Not homogenous

$\sin(y/x) = y/x - \frac{(y/x)^3}{3!} + \frac{(y/x)^5}{5!}$

homogeneous

$$\frac{dy}{dx} = v + \sec v$$

$$y = vx$$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$v + \sec v = \frac{dv}{dx}x + v$$

$$\frac{dv}{\sec v} = \frac{1}{x}dx$$

$$\int\cos v\,dv = \log xc$$

$$\sin v = \log xc$$

$$e^{\sin v} = xc$$

$$e^{\sin y/x} = xc$$

$$\sin y/x = \log xc$$

Exact D.Eq

→ The differential equation $\frac{dy}{dx} = f(x,y)$ $Mdx + Ndy = 0$ is said to be an exact diff eq if there exists a function $f(x,y)$ such that $d[f(x,y)] = Mdx + Ndy$

eg:- $y^2dx + 2xy\,dy = d(xy^2)$

Statement:-

→ The differential eq. $\boxed{Mdx + Ndy = 0 \text{ is exact} \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Solution

$$\boxed{\underset{\text{taking } y \text{ as constant}}{\int M\,dx} + \int(\text{terms of } N \text{ without } x)\,dy = c}$$

Note $Mdx + Ndy = 0$

$\Rightarrow d(f(x,y)) = 0 \Rightarrow \boxed{f(x,y) = C}$

eg: 7 General sol. of $(3x^2y^2 + x^2)\,dx + (2x^3y + y^2)\,dy = 0$

Solution $f(x,y) = x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} = c$

$\frac{\partial M}{\partial y} = 6x^2y = \frac{\partial N}{dx} = 6x^2y$

$\int M\,dx + \int N_{y'}\,dy = x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} = c$

eg:- $(x^4 - ax^2y^2 + y^4)\,dx + (bx^3y - bxy^3)\,dy = 0$ is exact then find a & b.

$\frac{\partial M}{\partial y} = -2ax^2y + 4y^3 = \frac{\partial N}{\partial x} = 3bx^2y - by^3$

$\Rightarrow$ $b = -4$ $a = \cancel{}\,6$

Non Exact Diff. Eq

→ An Non Exact Diff. Eq is converted to Exact by multiplying with a function $f(x,y)$ then $f(x,y)$ is called integrating factor

eg- $\boxed{y\,dx - x\,dy = 0}$

↓ IF

$\frac{1}{y^2}$	$\frac{1}{x^2}$	$\frac{1}{xy}$	$\frac{1}{x^2+y^2}$	$\frac{1}{y^2-x^2}$

Note

→ any constant multiple of an integrating factor is also an integrating factor

Finding I.F

Type I : All the terms of M & N should be of same degree

$$\boxed{IF = \frac{1}{Mx+Ny}}$$ $Mx + Ny \neq 0$

Type II : $yf(xy)\,dx + xg(xy)\,dy = 0$

$$\boxed{IF = \frac{1}{Mx - Ny}}$$

Type III : $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ only or constant

$$\boxed{IF = e^{\int f(x)\,dx}}$$

Type IV : $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ only or constant

$$\boxed{IF = e^{\int g(y)\,dy}}$$

Type V : Inspection method.

$\boxed{ydx - xdy}$

$$\boxed{\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)}$$

$$\boxed{\frac{ydx - xdy}{x^2} = d\left(\frac{-y}{x}\right)}$$

$$\boxed{\frac{ydx - xdy}{xy} = d\left(\log x/y\right)}$$

$$\boxed{\frac{ydx - xdy}{x^2+y^2} = d\left(\tan^{-1} x/y\right)}$$

$$\boxed{\frac{ydx - xdy}{y^2-x^2} = d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right]}$$

eg.) $ydx - x(1-y^2)dy = 0$

$ydx - xdy + xy^2dy = 0$

$\Rightarrow \dfrac{ydx - xdy + xy^2dy}{xy} = 0$

$\int d\left(\log \dfrac{x}{y}\right) + \int y\,dy \Rightarrow \log\left(\dfrac{x}{y}\right) + \left(\dfrac{y^2}{2}\right) = c$

$\boxed{xdx + ydy}$

$$\boxed{\frac{xdx + ydy}{x^2+y^2} = d\left[\tfrac{1}{2}\log(x^2+y^2)\right]}$$

$$\boxed{\frac{xdx + ydy}{(x^2+y^2)^2} = d\left[\frac{-1}{2}\left(\frac{1}{x^2+y^2}\right)\right]}$$

$\boxed{ydx + xdy}$

$ydx + xdy = d(xy)$

$\dfrac{ydx + xdy}{xy} = d\left[\log(xy)\right]$

$\dfrac{ydx + xdy}{x^2y^2} = d\left[\dfrac{-1}{xy}\right]$

$\dfrac{ydx + xdy}{x^3y^3} = d\left[-\dfrac{1}{2}\left[\dfrac{1}{x^2y^2}\right]\right]$

eg 8) Solution of $(x^2y^2+y)dx+(2x^3y-x)dy=0$

$x^2y^2dx+2x^3y\,dy = x\,dy-y\,dx$

$x^2[y^2dx+2xy\,dy] = x\,dy-y\,dx \Rightarrow y^2dx+2xy\,dy = d(y/x)$

Common terms outside

$= d(xy^2) = d(y/x) \Rightarrow xy^2 = \frac{y}{x}+c \Rightarrow$ ~~$x^2 = \frac{1}{y}$~~

$xy^2 - \frac{y}{x} = c$

eg 9) $(y-xy^2)dx+(x+x^2y)dy=0$

$y\,dx+x\,dy = xy^2dx - x^2y\,dy$

$y\,dx+x\,dy = xy(y\,dx - x\,dy)$

$\boxed{\frac{y\,dx+x\,dy}{(xy)^2} = \frac{y\,dx-x\,dy}{xy} = \cancel{d(xy)}}$

$\cancel{d(\log xy) = d(xy)} \Rightarrow \cancel{\log(xy)}$

$\Rightarrow d\left[\frac{-1}{xy}\right] = d\left(\log\frac{x}{y}\right)$

$\frac{-1}{xy} = \log\left(\frac{x}{y}\right)+c$

eg 10) Solution of $2xy^3dx+(3x^2y^2+x^2y^3+1)dy=0$ (Note type IV)

$\frac{\partial M}{\partial y} = 6xy^2 \qquad \frac{\partial N}{\partial x} = 6xy^2+2xy^3$

$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{6xy^2+2xy^3-6xy^2}{2xy^3} = 1$

$IF = e^{\int 1\,dy} = e^y$

$2xy^3e^y dx + (3x^2y^2+x^2y^3+1)e^y dy = 0$

solution

$\int 2xy^3e^y dx + \int e^y dy = [x^2y^3e^y+e^y] = c$

$\Rightarrow e^y(x^2y^3+1) = c$

eg 12.) $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$

$$x^2[ydx - xdy] - 2xy^2dx + 3x^2y\,dy = 0$$

$$ydx - xdy - \frac{2}{x}y^2dx + 3y\,dy = 0$$

$$\frac{ydx - xdy}{y^2} - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

$$\Rightarrow d\left[\frac{x}{y}\right] - \frac{2}{x}dx + \frac{3}{y}dy = 0$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} + \log\left(\frac{y^3}{x^2}\right) = c$$

$$x(ydx + xdy)\cos(y/x) = y(xdy - ydx)\sin(y/x)$$

$$\frac{ydx + xdy}{y} = \frac{xdy - ydx}{x}\tan(y/x) \Rightarrow \frac{ydx + xdy}{xy} = \frac{xdy - ydx}{x^2}\tan(y/x)$$

$$\Rightarrow d(\log(xy)) = \tan(y/x)\,d(y/x) \Rightarrow \log(xy) = \log\left|\sec\frac{y}{x}\right| + \log$$

$$\Rightarrow xy = C\sec\frac{y}{x}$$

---

eg 11.) $r\sin\theta\,d\theta + (r^3 - 2r^2\cos\theta + \cos\theta)dr = 0$

$$\frac{\partial M}{\partial r} = \sin\theta \qquad \frac{\partial N}{\partial \theta} = +2r^2\sin\theta - \sin\theta$$

Difference $2r^2\sin\theta - 2\sin\theta$

$$\frac{\frac{\partial N}{\partial \theta} - \frac{\partial M}{\partial r}}{M} = \frac{2r^2\sin\theta - 2\sin\theta}{r\sin\theta} = \left(2r - \frac{2}{r}\right)$$

$$I.F = e^{\int(2r - \frac{2}{r})dr} = e^{r^2 - 2\log r} = e^{r^2 + \log(r^{-2})} = e^{r^2}\cdot r^{-2} = \frac{e^{r^2}}{r^2}$$

Solution

$$\int r\sin\theta\cdot\frac{e^{r^2}}{r^2}d\theta + \cancel{\int\frac{\cos\theta\, e^{r^2}}{r^2}dr} = \frac{e^{r^2}}{r}(-\cos\theta) = C$$

$$\int re^{r^2}dr$$

$r^2 = t$

$2r\,dr = dt$

$$-\frac{e^{r^2}}{r}\cos\theta + \left[\cancel{re^{r^2}} - \cancel{e^{r^2}}\int\frac{e^t}{2}dt\right]$$

$$= -\frac{e^{r^2}}{r}\cos\theta + \frac{1}{2}e^{r^2} = c$$

Linear Diff. Eq

→ A diff Eq is said to be a linear diff eq if the dependent variable and all its derivatives are of first degree only.

→ They should not exist as products. ie all coefficient of the y and its derivatives should be functions of x only or constants.

$$y + a_1 y' + a_2 y'' + \dots a_{n-1} y^n = b_n$$

$a_1 \dots a_{n-1}$ constants or functions of x.

Note

→ Any differential eq. containing function of dependent variable is not linear in that variable.

eg: $(x+y)\frac{dy}{dx} = 1$ non linear in y but $(x+y) = \frac{dx}{dy}$ linear in x

Linear in y

$$\boxed{\frac{dy}{dx} + Py = Q}$$ P & Q are functions of x alone

Solution

$$\boxed{y e^{\int P dx} = \int Q e^{\int P dx} dx + C}$$

$$\boxed{e^{\int P dx} = \text{Integrating factor}}$$

Linear in x

$$\boxed{\frac{dx}{dy} + Px = Q}$$ P, Q are functions of y alone

Solution

$$\boxed{x e^{\int P dy} = \int Q e^{\int P dy} dy + c}$$

$$\boxed{e^{\int P dy} = I.F}$$

$$\frac{dy}{dx} = Q - Py \Rightarrow dy + (Py - Q)dx = 0$$

$$\frac{\partial M}{\partial y} = P \qquad \frac{\partial M}{\partial x} = 0$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{1} = \frac{P}{1}$$

$$\boxed{I.F = e^{\int P dx}}$$

solution

$$e^{\int P dx} dy = Q e^{\int P dx} \cdot dx \Rightarrow y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

* $\frac{dy}{dx} + \frac{2}{x}y = 3 \Rightarrow P = \frac{2}{x} \quad Q = 3$

$$y e^{2\int \frac{1}{x}dx} = yx^2 = \int 3x^2 dx + C \Rightarrow yx^2 = x^3 + C$$

---

eg 13) $x^4 y' + 4x^3 y = x^8 \Rightarrow P = \frac{4}{x} \quad Q = x^4$

$$\frac{dy}{dx} + \frac{4}{x}y = x^4$$

$$yx^4 = \frac{x^9}{9} + C$$

$$y = \frac{x^5}{9} + \frac{C}{x^4}$$

eg 14) $\frac{dy}{dx} + y = \sin x$

$y(\pi) = 1$

$$ye^x = \int e^x \sin x \, dx + C$$

$$ye^x = \frac{e^x}{2}(\sin x - \cos x) + C$$

$$e^\pi = \frac{e^\pi}{2}(+1) + C$$

$$C = \frac{e^\pi}{2}$$

$$\Rightarrow ye^x - \frac{e^x}{2}(\sin x - \cos x) = \frac{e^\pi}{2}$$

$$y = \frac{e^{\pi - x}}{2} + \frac{1}{2}(\sin x - \cos x)$$

if not linear in y also check for linear in x

* $\frac{dy}{dx} = \frac{1}{e^{-y} - x}$ at $y = 0, x = 1$

$$e^{-y} - x = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} + x = e^{-y}$$

$P = 1 \quad Q = e^{-y}$

$$xe^{y} = \int e^{-y} e^{y} dy + C$$

$$xe^y = \cancel{(e^{x-y})}\, y + C$$

$$e^1 = \cancel{(e^1)}\, y + C$$

$$C = 1$$

$$\Rightarrow \cancel{x = \frac{-e^{x-y} + 2e^{1-x}}{e^x} \Rightarrow x = -e^{-y} + 2e^{(1-x)}}$$

$$x = (y+1)e^{-y}$$

Non lineas equations of the form $\boxed{f'(y)\frac{dy}{dx} + P f(y) = Q}$ P & Q are functions of only x or constant.

$f(y) = V$

$f'(y) \cdot \frac{dy}{dx} = \frac{dV}{dx}$

$\Rightarrow \boxed{\frac{dV}{dx} + P \cdot V = Q}$ lineas in V

eg 16) $\tan y \left(\frac{dy}{dx}\right) + \tan x = \cos y \cos^2 x$

$\sec y = V$

$\sec y \tan y \frac{dy}{dx} = \frac{dV}{dx}$

$\frac{\tan y}{\cos y}\left(\frac{dy}{dx}\right) + \frac{\tan x}{\cos y} = \cos^2 x$

Hint to identify sub

P is function of x alone whatever is a product of function of x might be suitable

also making of Q function of x alone.

$\sec y \tan y \left(\frac{dy}{dx}\right) + \sec y (\tan x) = \cos^2 x$

$\frac{dV}{dx} + V \tan x = \cos^2 x$ $\quad P = \tan x$, $Q = \cos^2 x$

$\Rightarrow V e^{\int \tan x\, dx} = \int \cos^2 x\, e^{\int \tan x\, dx} + c$

$\Rightarrow V|\sec x| = \int \cos^2 x \sec x + c$

$V \sec x = \sin x + c$

$\sec y \sec x = \sin x + c$

$\sec y = (\sin x + c)\cos x$

* The substitution $y^{1-n} = V$ reduces the non lineas eq. $\boxed{\frac{dy}{dx} + Py = Qy^n}$ to which lineas form:

$y^{-n}\frac{dy}{dx} + P y^{(1-n)} = Q$

$y^{(1-n)} = V$

$\frac{dV}{dx} = (1-n) y^{-n} \times \frac{dy}{dx}$

$\frac{1}{(1-n)}\frac{dV}{dx} + PV = Q$

$\frac{dV}{dx} + P(1-n)V = Q(1-n)$

eg 15) $\frac{dy}{dx} + \frac{3y}{2} = \frac{3xy^{1/3}}{2}$

$$y^{-1/3}\frac{dy}{dx} + \frac{3}{2}y^{2/3} = \frac{3x}{2}$$

$$\frac{3}{2}\frac{dv}{dx} + \frac{3}{2}v = \frac{3x}{2}$$

$$\frac{dv}{dx} + v = x$$

$$\boxed{\begin{gathered} y^{2/3} = v \\ \frac{2}{3}y^{-1/3}\frac{dy}{dx} = \frac{dv}{dx} \\ P = 1 \\ Q = x \end{gathered}}$$

$$ve^{x} = \int xe^{x}\,dx$$

$$ve^{x} = xe^{x} - e^{x} + c$$

$$v = x - 1 + ce^{-x} \Rightarrow \underline{y^{2/3} = x - 1 + ce^{-x}}$$

Higher order linear differential equations with constant coefficients

Differential operator $\left(D = \frac{d}{dx}\right)$

eg $D^2 = \frac{d^2}{dx^2}$

eg $D^2(e^{3x}) = 9e^{3x}$

Inverse differential operator

$$\frac{1}{D} = \int dx$$

eg $\frac{1}{D^3} = \iiint dx\,dx\,dx$

eg $\frac{1}{D^2}(e^{4x}) = \frac{e^{4x}}{16}$

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + k_{n-1}\frac{dy}{dx} + k_n y = Q$$

$$\boxed{\begin{gathered} (D^n + k_1D^{n-1} + \dots k_{n-1}D + k_n)y = Q \\ F(D)y = Q \text{ — linear differential equation with constant coefficient} \end{gathered}}$$

$k_1 \dots k_n$ are constants

Q - function of x

Let the given differential equation is $F(D)y = Q$

→ ① The complete solution of the given equation is $y = C.F + P.I$

→ ② If $Q = 0$, then $F(D)y = 0$ is called homogeneous linear Diff. Eq

→ ③ If $Q \neq 0$, then $F(D)y = Q$ is called non-homogeneous linear diff. eq.

→ ④ The solution of homogeneous Linear Diff. Eq $F(D)y = 0$ is called the Complimentary Function (C.F)

→ ⑤ The no. of arbitrary constants in the complementary function should be equal to the ~~degree~~ order the given diff. eq.

⑥ → Particular Integral (P.I) of the given equation is

$$\boxed{P.I = \frac{1}{F(D)}[\phi]}$$

⑦ → If $\phi = 0$ then the complete solution is only C.F

⑧ → By assumming D as an algebraic quantity $F(D)=0$ becomes an algebraic equation and is known as the <u>auxiliary equation</u> of the given differential equation.

<u>Procedure to Find the Complementary Function</u>

⟶ By solving the auxiliary equation we get the roots. Based on the nature of these roots, we write the complementary function as follows.

Roots	Complimentary Function
→ Real & Distinct $D = m_1, m_2, m_2$	$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
→ Real & Repeated $D = m_1, m_2$	$C.F = (c_1 + c_2 x) e^{m_1 x}$
→ Complex & distinct $D = a \pm ib$	$C.F = e^{ax}[c_1 \cos b_1 x + c_2 \sin bx]$ $C.F = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$
→ Complex & Repeated $D = a \pm ib, a \pm ib$	$C.F = e^{ax}[(c_1 + c_2 x)\cos bx + (c_3 + c_4 x) \sin bx]$ $C.F = (c_1 + c_2 x) e^{(a+ib)x} + (c_3 + c_4 x) e^{(a-ib)x}$
→ surds $D = a \pm \sqrt{b}$	$C.F = c_1 e^{(a+\sqrt{b})x} + c_2 e^{(a-\sqrt{b})x}$ $C.F = e^{ax}[c_1 \cosh \sqrt{b} x + c_2 \sinh \sqrt{b} x]$

eg:-

Roots	C.F
$D=1,-2,-2$	$y=c_1e^{x}+(c_2+c_3x)e^{-2x}$
$D=2\pm i3, \frac{-1}{2}$	$y=e^{2x}[c_1\cos 3x+c_2\sin 3x]+c_3e^{-1/2x}$
$D=\pm 2i, \pm 4$	$y=c_1e^{4x}+c_2e^{4x}+[c_1\cos 2x+c_2\sin 2x]$

* $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

$F(D)=D^2-5D+6=0$

$\cancel{(D+6)(D+1)}$

$(D-3)(D-2)=0$

$D=2,3$

$y=c_1e^{2x}+c_2e^{3x}$

* $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

$F(D)=D^3+D^2+D+1=0$

$\Rightarrow D^2(D+1)(D+1)=0$

$(D+1)(D^2+1)=0$

$D=-1, \pm i$

$y=c_1e^{-x}+c_2\cos x+c_3\sin x$

Note *
→ Find first root by trial zeros
→ They factorise by division.

* $\frac{d^4y}{dx^4}-81y=0$

$\Rightarrow D^4-81=0$

$D^4=81$

$(D^2-9)(D^2+9)=0$

$D=\pm i3, \quad D=\pm 3$

$D^4=81$
$D^2=\pm 9$
$D=\sqrt{9}, \sqrt{-9}$
$D=\pm 3, \pm i3$

$y=c_1e^{3x}+c_2e^{-3x}+c_3\cos 3x+c_4\sin 3x$

* $\frac{d^2y}{dx^2}+\frac{R}{L}\frac{dy}{dx}+\frac{y}{LC}=0$

$R^2C=4L$

$D^2+\frac{R}{L}D+\frac{1}{LC}=0$

$\frac{-\frac{R}{L}\pm\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}}{2}=\frac{-\frac{R}{L}\pm\sqrt{\frac{R^2C-4L}{L^2C}}}{2}$

$=\frac{-\frac{R}{L}\pm 0}{2}=-\frac{R}{2L}, -\frac{R}{2L}$

$y=(c_1+c_2x)e^{-\frac{R}{2L}x}$

Note: if $y=c_1y_1+c_2y_2+c_3y_3+\cdots$ is the complete solution of the homogeneous linear differential equation $f(D)y=0$, then each one of $y_1, y_2, y_3, \ldots$ so no. are linearly independent solutions of the same homogeneous linear diff. eq. $F(D)y=0$

$\boxed{\text{N.B ONLY HOMOGENEOUS}}$

eg) $\sin 2x, \cos 2x, e^x$ are the linearly independent solutions of which of the following differential eq.

roots $= \pm 2i, 1$

$F(D) = (D^2+4)(D-1)$

$F(D)y = (D^3 - D^2 + 4D - 4)y = 0$

eg 23)

$c_1 e^x + e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}x}{2}\right]$ is general solution of ?

$D = 1, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$ $\quad \frac{1}{2}+$

$(D-1)\left(D + \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right) = (D-1)\left(D^2 + \frac{1}{4} + D + \frac{3}{4}\right)$

$= D^3 - \frac{D}{2} + D^2 - D^2 + \frac{1}{2} - D = 0$

~~$(D-1)\left(D^2 + \frac{1}{4} - 2D - \frac{3}{4}\right)$~~ $\Rightarrow D^3 - \frac{3}{2}D = -\frac{1}{2}$

~~$D^3 + \frac{D}{4} - D^2 - \frac{3}{4}D - D^2 + \frac{2}{4} - D = 0$~~

eg 3)

$y = c_1 \cos x + c_2 \sin x$ is solution of

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

then the solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + (Q-1)y = e^x$ is $= ?$

$D = \pm i \quad (D^2+1) = 0$

$P = 0$

$Q = 1$

$\frac{d^2y}{dx^2} = e^x$

$d^2y = e^x\, dx^2$

$y = e^x + cx + c$

during direct integration c must not be omitted for C.F

eg 17) General solution of $y'' + 4y' + 5y = 0$

$D^2 + 4D + 5 = 0$ $\Rightarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x)$

$D = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

eg 18) $y'' - 4y' - 6y = 0$ $\quad D^2 - 4D - 6 = 0$

$D = \frac{4 \pm \sqrt{16+24}}{2} = 2 \pm \sqrt{10}$

$y = e^{2x}\left(c_1 \sinh(\sqrt{10})x + c_2 \cosh(\sqrt{10})x\right)$

eg 19) $y''' - 6y'' + 11y' - 6y = 0$

$$D^3 - 6D^2 + 11D - 6 = 0$$

$$
\begin{array}{r|l}
 & D^2 - 5D + 6 \\
\hline
D-1 & D^3 - 6D^2 + 11D - 6 \\
 & D^3 - D^2 \\
\hline
 & -5D^2 + 11D \\
 & -5D^2 + 5D \\
\hline
 & 6D - 6 \\
 & 6D - 6 \\
\hline
 & 0
\end{array}
$$

$(D-1)\ (D-2)\ (D-3)$

$D = 1, 2, 3$

$$\underline{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}$$

20.) $(D^2+1)^2 = 0$

$D^2 = -1 \quad D = \pm i, \pm i$

$$\underline{y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x}$$

21) $D^2 = -4 \Rightarrow D = \pm 2i$ | $y(0) = 0$, $y'(0) = 10$

$y = c_1 \cos 2x + c_2 \sin 2x$

$0 = c_1$

$y' = -2c_1 \sin 2x + 2c_2 \cos 2x$

$10 = 2c_2 \Rightarrow c_2 = 5$

$\therefore y = 5\sin 2x \Rightarrow y(1) = \underline{\underline{5\sin(2)}}$

22) $\frac{d^2y}{dx^2} + y = 0 \Rightarrow D = \pm i$

$y = c_1 \cos x + c_2 \sin x$

$y(0) = 1 \Rightarrow 1 = c_1$

$y'(0) = 0$

$y = \cos x + c_2 \sin x$

$y' = -\sin x + c_2 \cos x$

$y'(0) = 0 = c_2 \Rightarrow \underline{c_2 = 0}$

$\therefore y = \underline{\underline{\cos x}}$

23) $c_1 e^{x} + e^{-x/2}\left[\cos\left(\frac{\sqrt{3}}{2}\right)x + \sin\left(\frac{\sqrt{3}}{2}\right)x\right] = y$

$D = 1, \left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \Rightarrow (D-1)\left(D + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(D + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$\Rightarrow (D-1)\left(D^2 + \frac{1}{4} + D + \frac{3}{4}\right) \Rightarrow (D-1)(D^2 + D + 1)$

$\Rightarrow D^3 + \cancel{D^2} + \cancel{D} - \cancel{D^2} - \cancel{D} - 1 = 0 \Rightarrow D^3 - 1 = 0$

$$\therefore \underline{\underline{\frac{d^3y}{dx^3} - y = 0}}$$

24) $e^{x}, e^{x}\cos x, e^{x}\sin x$ are independent solutions $\therefore$ diff eq = ?

Sol $D = 1, (1 \pm i)$

$\therefore (D-1)\ (D-1+i)(D-1-i)$

$(D-1)\ (D^2 + 1 - 2D + 1) = 0$

$\Rightarrow (D-1)\ (D^2 - 2D + 2) = 0$

$\Rightarrow D^3 - 2D^2 + 2D - D^2 + 2D - 2 = 0$

$\Rightarrow D^3 - 3D^2 + 4D - 2 = 0$

$$\underline{y''' - 3y'' + 4y' - 2y = 0}$$

*) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0$ | $y(0) = y'(0) = 1$

$D = -2, -2$

$y = (c_1 + c_2 x)e^{-2x}$

$1 = c_1$

$y' = -2e^{-2x} - 2c_2 x e^{-2x} + c_2 e^{-2x}$

$1 = -2 + c_2$

$c_2 = 3$

$$\underline{\underline{y = (1 + 3x)e^{-2x}}}$$

$y(1) = (1+3)e^{-2} = \underline{4e^{-2}}$

* $D^2+2D+1=0$

$(D+1)^2=0$

$D = -1, -1$

$y = (c_1+c_2x)e^{-x}$

$y(0)=y'(0)=1$

$c_1e^{-0}=1 \Rightarrow c_1=1$

$y' = -e^{-x} + -c_2xe^{-x} + c_2e^{-x}$

$1 = -1 + c_2$

$\underline{c_2=2}$

$\therefore y = y = \underline{(1+2t)e^{-t}}$

---

* $\frac{d^2y}{dx^2}+16y=0 \quad \Big| \quad y'(0)=1 \;\&\; y'\left(\frac{\pi}{2}\right)=-1$

$D^2 = -16 \qquad D = \pm 4i$

$y = c_1\cos 4x + c_2 \sin 4x$

$y' = -4c_1\sin 4x + 4c_2\cos 4x$

$y'(0) = 1 = 4c_2 \Rightarrow c_2 = 1/4$

$y'\left(\frac{\pi}{2}\right) = -1 = \cancel{-4c_1}\; +4c_2 \Rightarrow c_2 = -1/4$

$\Rightarrow y = \cancel{\frac{1}{4}(\cos 4x + \sin 4x)}$

<u>No solution</u>

---

* $\frac{d^2n}{dt^2} - \frac{n}{L^2} = 0 \quad \Big| \quad n(0)=k,\; n(\infty)=0$

$D^2 - \frac{1}{L^2} = 0$

$D = \pm\frac{1}{L}$

$n = c_1e^{-1/L\,x} + c_2e^{1/L\,x}$

$k = c_1 + c_2$

$0 = c_2\infty \Rightarrow c_2 = 0$

$\Rightarrow c_1 = k$

$\therefore n = \underline{ke^{-1/L}}$

---

* $y''=0 \quad \Big| \quad y'(0)=1,\; y'(1)=1$

$y' = c$

$\Rightarrow y = c_1x + c_2$

$y' = c_1 \Rightarrow y'(0) = 1 = c_1$, $y'(1) = 1 = c_1 \Rightarrow c_1 = 1$

$\therefore y = x + c_2$ <u>infinite solution</u>

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* D.Eq $D^3 + xD^2 - D + 3 = x^4$ is?

(a) Homo. Linear Diff Eq

(b) Non Hom Linear Diff Eq with constant coefficient.

(c) Non linear D.Eq of order 3

✓ (d) Linear differential equation of order 3.

<u>Particular Integrals</u>

$P.I = \left[\frac{1}{F(D)}\right]\phi \quad \Big| \quad F(D)y = \phi$

ϕ:

- $\boxed{e^{ax} \text{ or constant}}$
  - $\frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}$, $F(a) \neq 0$
  - if $F(a) = 0$, $PI = \frac{x}{F'(D)}e^{ax}$
- $\boxed{\sin ax \text{ or } \cos bx}$
- $\boxed{x^m \text{ or poly in 'x'}}$
- $\boxed{e^{ax}.V}$
  - $\boxed{-\sin ax \;\; -\cos bx}$
  - $\boxed{-x^m \;\; -\text{poly in 'x'}}$
- $\boxed{x.V}$
  - $\boxed{-\sin ax \;\; -\cos bx}$

$$\boxed{\left(\frac{1}{D-\alpha}\right)X = e^{\alpha x}\int X e^{-\alpha x}\,dx}$$

eg. $\frac{dy}{dx} - 2y = x^2$ $\Rightarrow D-2=0 \Rightarrow D=2$ $\quad C.F = c_1 e^{2x}$

$$P.I = \left(\frac{1}{(D-2)}\right)x^2 = e^{2x}\int x^2 e^{-2x}dx = e^{2x}\left[\frac{x^2 e^{-2x}}{-2} - \frac{2xe^{-2x}}{4} + \frac{2e^{-2x}}{-8}\right]$$

$$P.I = \frac{x^2}{-2} - \frac{2x}{4} + \frac{2}{-8} = \frac{x^2}{-2} - \frac{x}{2} + \frac{1}{-4}$$

$$y = c_1 e^{2x} - \frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$$

$*$ P.I of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

$$P.I = \frac{1}{(D^2-5D+6)}(x^2) = \frac{1}{(D-3)(D-2)}(x^2)$$

$$= \left(\frac{1}{D-3} - \frac{1}{(D-2)}\right)x^2$$ do partial fraction

$$= \left(\frac{1}{D-3}\right)x^2 - \left(\frac{1}{D-2}\right)x^2$$ {little bit easy}

$\Rightarrow \frac{1}{(D-2)}x^2 = -\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}$ {previous question}

$$\frac{1}{D-3}\left\{-\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right\}$$

$$= e^{3x}\int\left\{-\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right\}e^{-\alpha x}dx$$ {very lengthy}

Case I $\quad X = e^{ax}$ or constant $\quad$ replace 'D' by coefficient 'a'

$$P.I = \left(\frac{1}{F(D)}\right)e^{ax} = \frac{1}{F(a)}e^{ax} \quad \left(\text{provided } F(a)\neq 0\right)$$

if $F(a)=0$ then $P.I = \left[\frac{x}{F'(D)}\right]e^{ax} = \frac{x}{F'(a)}e^{ax}$ provided $F'(a)\neq 0$

if $F'(a)=0$ then $P.I = \left[\frac{x^2}{F''D}\right]e^{ax} = \frac{x^2 e^{ax}}{F''(a)}$ provided $F''(a)\neq 0$

Note: the solution will always contain e^{ax}

$*$26) P.I $\quad y^{iv} + 4y''' + 8y'' + 8y' + 4y = 20$

$$\left(\frac{1}{D^4+4D^3+8D^2+8D+4}\right)20e^{0x} = \frac{20}{4} = 5$$

$*$27.) P.I of $\frac{d^5y}{dx^5} - \frac{dy}{dx} = 12e^x$ $\Rightarrow$ $P.I = \left(\frac{1}{D^5-D}\right)12e^x = \frac{x\,12e^x}{5D^4-1} = \frac{12xe^x}{4}$

$$P.I = 3xe^x$$

28) P.I of $\frac{d^2y}{dt^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$

$$P.I = \left(\frac{1}{D^2+4D+5}\right) -2\left(\frac{e^{-x}+e^{x}}{2}\right) = -\frac{e^{-x}}{1-4+5} - \frac{e^{x}}{1+4+5} = \frac{-e^{-x}}{2} - \frac{e^{x}}{10}$$

$$P.I = \frac{-1}{2}\left(e^{-x} + \frac{e^{x}}{5}\right)$$

* P.I of $4y'' - 4y' + y = e^{x/2}$

$$\left(\frac{1}{4D^2-4D+1}\right)e^{x/2} = \left(\frac{x}{8D-4}\right)e^{x/2} = \frac{x^2 e^{x/2}}{8}$$

Case II $Q = \sin ax$ or $\cos ax$ or $X = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I = \left(\frac{1}{F(D)}\right)\sin(ax)$$

Procedure : replace D^2 by $-(a)^2$

$$\rightarrow \left(\frac{1}{F(D)}\right)\sin ax = \left(\frac{1}{F(D^2=-a^2)}\right)\sin ax \quad \text{provided } F(D^2=-a^2) \neq 0$$

if $F(D^2=-a^2) = 0$ then

$$\left(\frac{1}{F(D^2)}\right)\sin ax = \left(\frac{x}{F'(D)}\right)\sin ax = x\left(\frac{1}{F'(D^2=-a^2)}\right)\sin ax \quad \text{provided } F'(D^2=-a^2) \neq 0$$

$$\rightarrow \left(\frac{1}{a+bD}\right)\sin ax = \left(\frac{a-bD}{a^2-b^2D^2}\right)\sin ax$$

* P.I of $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} - 2y = \cos(2x+4)$

$$P.I = \left[\frac{1}{D^4+D^2-2}\right]\cos 2x+4 = \left(\frac{1}{16-4-2}\right)\cos(2x+4) = \frac{\cos(2x+4)}{10}$$

* P.I of $y'' + 16y = e^{-4x} + \cos 4x$

$$P.I = \left[\frac{1}{D^2+16}\right]e^{-4x} + \left[\frac{1}{D^2+16}\right]\cos 4x = \left[\frac{e^{-4x}}{32}\right] + \left(\frac{\cancel{x}\cos 4x}{\cancel{-10}}\right)$$

$$\therefore PI = \frac{e^{-4x}}{32} + \cancel{\frac{\cos 4x}{-16}} \; \cancel{\frac{1}{2}\left[x\frac{e^{-4x}}{-4}\right.} \; \cancel{+ e^{-4x}} + x\left(\frac{1}{2D}\right)(\cos 4x)$$

$$= \frac{e^{-4x}}{32} + \frac{x \sin 4x}{8}$$

Note $\boxed{x \text{ is not to be operated}}$

eg. 32) P.I of $y''' + y = \sin 3x$

$$P.I \left(\frac{1}{D^3+1}\right)\sin 3x = \left(\frac{1}{-9D+1}\right)\sin 3x$$

$$\left(\frac{1+9D}{1^2-81D^2}\right)\sin 3x = \frac{(1+9D)}{1^2+(81\times 9)}\sin 3x$$

$$= \frac{1+9D}{730}\sin 3x = \frac{1}{730}\sin 3x + \frac{27\cos 3x}{730}$$

$$P.I = \frac{\sin 3x + 27\cos 3x}{730}$$

Case III $\phi = x^m$ or polynomial in 'x'

note: (m = positive integer)

$$P.I = \left[\frac{1}{F(D)}\right]x^m = \left(\frac{1}{*[1+\phi(D)]}\right)x^m$$

where * is the term with least power of D

take that as common from denominator

then,

$$P.I = \frac{1}{*}[1+\phi(D)]^{-1}x^m$$

if * has D in it then cancel with expansion of numerator before applying to function or some terms will go missing in differentiation

Note

$$[1+\phi(D)]^{-1} = 1-\phi(D)+\phi^2(D)-\phi^3(D)+\ldots$$

$$[1-\phi(D)]^{-1} = 1+\phi(D)+\phi^2(D)+\phi^3(D)+\ldots$$

33) * P.I of $y''' + 8y = x^4 + 2x + 1$

$$P.I = \left(\frac{1}{D^3+8}\right)x^4+2x+1 = \frac{1}{8\left(1+\frac{D^3}{8}\right)}(x^4+2x+1)$$

$$= \frac{1}{8}\left(1-\frac{D^3}{8}+\frac{D^6}{64}\right)(x^4+2x+1) = \frac{1}{8}\left[x^4+2x+1-\frac{D^2}{8}(4x^3+2)\right]$$

$$= \frac{1}{8}\left[x^4+2x+1-\frac{D}{8}(12x^2)\right] = \frac{1}{8}\left[x^4+2x+1-\frac{24x}{8}\right] = \cancel{\frac{1}{8}\left[x^4+2x-2\right]} \quad \frac{1}{8}\left[x^4-x+1\right]$$

34) * P.I of $y'' - 4y' - 2y = x^2$

$$P.I = \left(\frac{1}{D^2-4D-2}\right)x^2 = \frac{1}{-2\left(-\frac{D^2}{2}+2D+1\right)}x^2$$

$$= \frac{1}{-2}\left[1-\left(\frac{D^2}{2}-2D\right)\right]^{-1}x^2 = \frac{1}{-2}\left[1+\left(\frac{D^2}{2}-2D\right)+\left(\frac{D^2}{2}-2D\right)^2\right]x^2 = \frac{1}{-2}\left[1+\frac{D^2}{2}-2D+4D^2\right]x^2$$

$$= \frac{1}{-2}\left[x^2+1-4x+8\right] = \frac{1}{-2}\left[x^2-4x+9\right]$$

CASE IV $\phi = e^{ax}\cdot V$ ($V = \sin bx$ or $\cos bx$; $V = x^m$ or polynomial in x)

$$P.I = \frac{1}{F(D)}(e^{ax}\cdot V)$$ replace 'D' by 'D+a' in F(D) to take e^{ax} outside

$$P.I = e^{ax}\left(\frac{1}{F(D+a)}\right)V$$

→ Case II if $V = \sin(ax)$ or $\cos bx$

→ Case III if $V = x^m$ or poly in "x"

eg 36) P.I of $y'' - 2y' + 5y' = e^{x} \cos 3x$ $\quad P.I = \left(\frac{1}{D^2 - 2D + 5}\right) e^{x} \cos 3x$

$P.I = e^{x}\left(\frac{1}{D^2 + 1 + 2D - 2D - 2 + 5}\right)\cos 3x = e^{x}\left(\frac{1}{D^2 + 4}\right)\cos 3x = e^{x} \cdot \frac{1}{(-9+4)}\cos 3x = \underline{\frac{e^{x}\cos 3x}{-5}}$

---

eg 35) $y'' + 2y' + y = x^2 e^{-x}$

$$P.I = \frac{1}{(D^2 + 2D + 1)} e^{-x} \cdot x^2 = e^{-x}\left(\frac{1}{((D^2 + 1 - 2D + 2D - 2 + 1)}\right) x^2 = e^{-x} \frac{1}{D^2}\left(x^2\right)$$

$$P.I = \underline{e^{-x} \cdot \frac{x^4}{12}}$$

Case V $\quad Q = x \cdot V \quad \underline{V = \sin bx \text{ or } \cos bx}$

$$P.I \left[\frac{1}{F(D)}\right] x \cdot V = x\left[\frac{1}{F(D)}\right] V - \left\{\frac{F'(D)}{[F(D)]^2}\right\} V$$

---

eg 37) $y'' + 4y = x \sin x$

When x and e^{ax} comes together use previous form & not this.

$$\frac{1}{(D^2 + 4)} x \sin x = x\left(\frac{1}{D^2 + 4}\right) \sin x - \left(\frac{2D}{D^4 + 8D^2 + 16}\right) \sin x$$

$$= x\left(\frac{1}{4-1}\right)\sin x - \frac{2\cos x}{1 - 8 + 16} = \underline{\frac{x \sin x}{3} - \frac{2}{9}\cos x}$$

Q30) $\frac{d^2y}{dx^2} + y = \cos x$ with $y(0) = 1$ & $y(\pi/2) = 0$

Aux Eq $= D^2 + 1 = 0 \quad D = \pm i$

$C.F = c_1 \sin x + c_2 \cos x$

$P.I = \left(\frac{1}{D^2 + 1}\right)\cos x = x\left(\frac{1}{2D}\right)\cos x$

$= \frac{x}{2} \cdot \sin x$

$y = c_1 \sin x + c_2 \cos x + \frac{x}{2} \sin x$

$y(\pi/2) = c_1 + \frac{\pi}{4} = 0$

$c_1 = -\pi/4$

$y(0) = 1 = \underline{c_2 = 1}$

$$\therefore y = \underline{\cos x - \frac{\pi}{4}\sin x + \frac{x \sin x}{2}}$$

*) $\frac{d^2y}{dx^2} + -4y = a^x$ $\boxed{\frac{1}{D^2-4} e^{x\log a}}$

★ $D^2 - 4 = 0$

$D = \pm 2$

$C.F = c_1 e^{2x} + c_2 e^{2x}.$

$P.I = \left[\frac{1}{D^2-4}\right] a^x$

$\frac{1}{(\log^2 a - 4)} e^{x\log a}$

$P.I = \frac{a^x}{\log^2 a - 4}$

$y = c_1 e^{2x} + c_2 e^{2x} + \frac{a^x}{\log^2 a - 4}$

* $\frac{d^2y}{dx^2} = e^x$ $\quad y(0) = 1$

$y'(0) = 2$

$y = e^x + c$

$y = e^x + c_1 x + c_2$

$y = x + e^x$

Note: clue
Solution must contain e^x

*) $k^2 \frac{d^2y}{dt^2} = y - y_2$; $y(0) = y_1$

$y(\infty) = y_2$

$\left(D^2 - \frac{1}{k^2}\right) = 0$

$D = \pm \frac{1}{k}$

$PI = \left(\frac{1}{D^2 - \frac{1}{k^2}}\right) \frac{-y_2}{k^2}$

~~PI = y2~~ $PI = y_2$

$y = c_1 e^{1/k\, x} + c_2 e^{-1/k\, x} + y_2$

$y(0) = y_1 = c_1 + c_2 + y_2$

$c_1 + c_2 = y_1 - y_2$

$y(\infty) = y_2 = c_1 \infty + c_2 \cdot 0 + y_2$

$c_1 = 0$

$c_2 = y_1 - y_2$

$\therefore y = (y_1 - y_2) e^{-1/k\, x} + y_2$

Method of variation of parameters

This is a method to find the P.I of $\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R}$

where P, Q are ~~functions~~ constants

R function of x or constant

note: only 2nd order

$\boxed{\begin{aligned} C.F &= c_1 y_1 + c_2 y_2 \\ P.I &= A y_1 + B y_2 \end{aligned}}$

$A = -\int \frac{R y_2}{W} dx$ $\quad B = +\int \frac{R y_1}{W} dx$

W = wronskian $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

*38) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$D^2 - 6D + 9 = 0 \Rightarrow (D-3)^2 = 0$

$C.F = (c_1 + c_2 x) e^{3x} = c_1 e^{3x} + c_2 x e^{3x}$

$W = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x} + 3x e^{3x} \end{vmatrix}$

$A = -\int \frac{e^{3x} \, x e^{3x}}{x^2 e^{6x}} = \underline{-\ln x}$

$P.I = \underline{-e^{3x} [\ln x + 1]}$

$B = \int \frac{e^{3x} e^{3x}}{x^2 e^{6x}} = -\frac{1}{x}$

$W = e^{6x} \cancel{+ 3x e^{6x} - 3x e^{6x}}$

$W = \underline{e^{6x}}$

Cauchy's Euler's D.Eq

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots k_{n-2} x^2 \frac{d^2y}{dx^2} + k_{n-1} x \frac{dy}{dt} + k_n y = Q$$

$k_1 \ldots k_n$ constants

Q – function in 'x'

$$\boxed{\begin{aligned} x &= e^t \\ \ln x &= t \end{aligned}}$$

$\cancel{\frac{dy}{dt}} =$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\boxed{x \frac{dy}{dx} = \frac{dy}{dt}}$$

$$\boxed{\begin{aligned} D &= \frac{dy}{dx} \\ \theta &= \frac{dy}{dt} \end{aligned}}$$

$$\boxed{\begin{aligned} xD &= \theta \\ x^2 D^2 &= \theta(\theta-1) \\ x^3 D^3 &= \theta(\theta-1)(\theta-2) \end{aligned}}$$

---

39) $x^2 y'' - 2xy' + 2y = 4$

$y(\theta(\theta-1) - 2\theta + 2) = 4$

$y(\theta^2 - \theta - 2\theta + 2) = 4$

$(\theta^2 - 3\theta + 2)y = 4$

$\theta = 2, 1$

$C.F = c_1 e^{2t} + c_2 e^{t}$

$P.I = \frac{1}{(\theta-2)(\theta-1)} 4 = \frac{4}{2}$

$\therefore y = c_1 e^{2t} + c_2 e^{t} + 4/2$

$y = c_1 x^2 + c_2 x + 4/2 = c_1 x^2 + c_2 x + 2$

---

40) solution of $x^2 y'' - 4xy' + 6y = 0$

$(\theta^2 - \theta - 4\theta + 6)y = 0$

$F(\theta) = (\theta^2 - 5\theta + 6) = 0$

$(\theta - 3)(\theta - 2) = 0$

$\theta = 3, 2$

$C.F = c_1 e^{3t} + c_2 e^{2t}$

$y = c_1 x^2 + c_2 x^3$

---

* $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$

$y(0) = 0$
$y(1) = 1$

$(\theta^2 - \theta + \theta - 4) = 0$

$(\theta^2 - 4) = 0$

$\theta = \pm 2$

$y = c_1 e^{-2x} + c_2 e^{2x}$

$y = c_1 x^2 + c_2 x^{-2}$

$c_2 = 0$
$c_1 = 1$
$y = x^2$

a) x^2
b) $\sin(x\pi/2)$
c) $e^x \sin(x\pi/2)$
d) $e^{-x} \sin(x\pi/2)$

Partial Differential Equations (PDE)

If $z = f(x,y)$ then we have the following.

$p = \frac{\partial f}{\partial x}$	$q = \frac{\partial f}{\partial y}$	$r = \frac{\partial^2 f}{\partial x^2}$	$s = \frac{\partial^2 f}{\partial x \partial y}$	$t = \frac{\partial^2 f}{\partial y^2}$

Formation of PDE

→ Elimination of arbitrary constants

→ If the no. of arbitrary constants eliminated are equal to the no. of independent variables, then the resulting partial diff. equations will be of first order.

→ If the no. of arbitrary constants are more than the no. of independent variables then the resulting P.D.E equations will be of higher order.

→ Elimination of arbitrary functions

→ The no. of arbitrary functions eliminated should be equal to the order of the resulting partial differential equation.

*41) a, b are arbitrary constants then the P.DE corresponding to $z = ax^n + by^n$

$$\frac{\partial z}{\partial x} = nax^{n-1} \qquad \frac{\partial z}{\partial y} = nby^{n-1}$$

$$\frac{p}{nx^{n-1}} = a \qquad \frac{q}{ny^{n-1}} = b$$

$$z = \frac{p}{nx^{n-1}}x^n + \frac{q}{ny^{n-1}}y^n = \frac{px}{n} + \frac{qy}{n}$$

$$nz = px + qy$$

42) The P.D.E obtained by eliminating the arbitrary function from the relation

$f(x^2+y^2+z^2, x+y+z) = 0$ is

$x^2+y^2+z^2 = f(x+y+z)$ ← Concept

① w.r.t x

$$2x + 2z\frac{\partial z}{\partial x} = f'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow 2x + 2zp = f'(x+y+z)(1+p)$$

$$\Rightarrow 2(x+zp) = f'(x+y+z)(1+p)$$

① w.r.t y

$$2(y+zq) = f'(x+y+z)(1+q)$$

$$(x+zp)(1+q) = (1+p)(y+zq)$$

$$x + xq + zp + \cancel{zpq} = y + zq + py + \cancel{pzq}$$

$$x - y = p(y-z) + q(z-x)$$

Classification of P.D.E

→ The P.D.E $A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$ is

1. parabolic if $B^2 - 4AC = 0$
2. hyperbolic if $B^2 - 4AC > 0$
3. elliptic if $B^2 - 4AC < 0$

43) $U_{xx} - 6U_{xy} + 9U_{yy} - xy^2 = 0$

$36 - (4 \times 1 \times 9) = 0 \quad \therefore$ parabolic

44) $3U_{xx} + 6U_{xy} - 16U_{yy} = 0$

$B = 6 \qquad 36 + 48 > 0$ hyperbolic

45) $6U_{xx} + 7U_{yy} - 3U_{xy} = 4U_x + U_y$

$B = -3$

$9 - 42 < 0$

elliptic

46) $x^5 U_{xx} - xU_{yy} + 2U_y = 0$

$4x^6 > 0$ hyperbolic

ONE DIMENSIONAL WAVE EQUATION

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

TWO DIMENSIONAL WAVE EQUATION

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

LAPLACE EQUATION (OR) TWO DIMENSIONAL HEAT EQ. UNDER STEADY STATE :

$$\boxed{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0}$$

ONE DIMENSIONAL HEAT EQUATION

$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

TWO DIMENSIONAL HEAT EQUATION

$$\boxed{\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

METHOD OF SEPARATION OF VARIABLES

48) Find the solution of P.D.E $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

$XT' = \alpha X''T \Rightarrow \frac{T'}{T} = \frac{\alpha X''}{X} = K$

$\Rightarrow \alpha \frac{d^2 X}{dx^2} = KX \Rightarrow \alpha \frac{d^2 X}{dx^2} - KX = 0$

$\Rightarrow \frac{dT}{dt} = KT \qquad \frac{dT}{dt} - KT = 0$

Solution $u = XT$

$X = f(x)$

$T = g(t)$

$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{\partial T}{\partial t} = X\frac{dT}{dt}$

$\frac{\partial u}{\partial t} = XT'$ similary

$\left.\begin{aligned} \frac{\partial u}{\partial x} &= X'T \\ \frac{\partial^2 u}{\partial t^2} &= T''X \\ \frac{\partial^2 u}{\partial x^2} &= X''T \\ \frac{\partial^2 u}{\partial x \partial t} &= X'T' \end{aligned}\right\}$ substitute in P.D.E

$\Rightarrow X\left(D^2 - \frac{k}{\alpha}\right) = 0$

$D = \pm\sqrt{\frac{k}{\alpha}}$

$X = c_1 e^{\sqrt{\frac{k}{\alpha}}x} + c_2 e^{-\sqrt{\frac{k}{\alpha}}x}$

$\Rightarrow T(D-k) = 0 \quad D = k$

$T = c_3 e^{kt}$

$u(x,t) = XT = \left(c_1 e^{\sqrt{\frac{k}{\alpha}}x} + c_2 e^{-\sqrt{\frac{k}{\alpha}}x}\right) c_3 e^{kt}$

---

49) $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ | $u(0,y) = 8e^{-3y}$

$\Rightarrow$ ~~$YX' = 4TY'$~~

$YX' = 4XY'$

$\frac{X'}{X} = 4\frac{Y'}{Y} = k$

$\Rightarrow \frac{dX}{dx} - kX = 0$

$(D-k) = 0$

$D = k$

$X = c_1 e^{kx}$

$\Rightarrow \frac{dY}{dy} - \frac{kY}{4} = 0$

$\left(D - \frac{k}{4}\right) = 0$

$Y = c_2 e^{+\frac{k}{4}y}$

$U = XY = c_1c_2 e^{kx + \frac{k}{4}y} \Rightarrow u(x,y) = ce^{kx+\frac{k}{4}y}$

$u(0,y) = 8e^{-3y} = ce^{+\frac{k}{4}y} \rightarrow c = 8$

$\frac{k}{4} = -3 \quad k = -12$

$\underline{\underline{y = 8e^{-12x-3y}}}$

Lagrange's Linear Differential Equations

An equation of the form $\boxed{Pp + Qq = R}$ is called Lagranges LDE

where P, Q and R are functions of x, y, z and $p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$

<u>Lagrange's Auxiliary Equations</u>

The auxiliary equations of $Pp + Qq = R$ are $\boxed{\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R}}$

Procedure to solve L's LDE

$\rightarrow$ Let the given equation is $Pp + Qq = R$

① Write the auxiliary equations of the given equation.

② Solve the auxiliary eq. in step 1 to get two independent solutions u & v by using grouping or multipliers method.

③ Complete solution of given equation is $f(u,v) = 0$ or $u = f(v)$ or $v = f(u)$.

Note

→ We select the multipliers P_1, Q_1, R_1 such that

① $P_1P + Q_1Q + R_1R = 0$

~~then~~ ② $P_1dx + Q_1dy + R_1dz$ should be integral

$$\frac{P_1dx + Q_1dy + R_1dz}{P_1P + Q_1Q + R_1R} = \frac{dy}{Q} = \frac{dx}{P} = \frac{dz}{R}$$

→ then by integrating $P_1dx + Q_1dy + R_1dz = 0$

we get the solution corresponding to the multipliers P_1, Q_1, R_1.

→ we can choose another pair multipliers P_2, Q_2, R_2

to similarly get another independent solution.

---

* The solution of $p - q = \log x+y$

50) $P = 1$

$Q = -1$

$R = \log x+y$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

$$\Rightarrow \frac{dx}{dy} = -1$$

$$x = -y + c \Rightarrow \boxed{x+y = c}$$

$$\frac{dx}{1} = \frac{dz}{\log c} = \frac{dx}{dz} = \frac{1}{\log c} \qquad x = cz + c$$

$$x - \frac{z}{\log(x+y)} = c$$

$$f\left(x+y,\ x - \frac{z}{\log(x+y)}\right) = 0$$

51) Solution of $(z-y)p + (x-z)q = y-x$ **Note No grouping works**

$$\frac{dx}{(z-y)} = \frac{dy}{(x-z)} = \frac{dz}{(y-x)}$$

$P_1 = Q_1 = R_1$

$\Rightarrow dx + dy + dz = 0$

$\Rightarrow x + y + z = c$

$P_1 = x$

$P_2 = y$ $\Rightarrow xdx + ydy + zdz = 0$

$P_3 = z$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$\therefore f(x+y+z,\ x^2+y^2+z^2) = 0$ $\rightarrow x^2+y^2+z^2 = c$

Note

Equations of the form $f(p,q)=0$. The solution of this is $z=ax+by+c$ where a & b are connected by the relation $f(a,b)=0 \Rightarrow b=f(a)$. Therefore $z=ax+f(a)y+c$.

52*) Find the solution of $q=3p^2$

$b=3a^2$

$z=ax+3a^2y+c$

Note

If the given equation is of the form $z=px+qy+f(p,q)$ then solution is obtained by replacing 'p' by 'a' & 'q' by 'b'.

solution $z=ax+by+f(a,b)$

55*) Solution of $(p-q)(z-px-qy)=1$

$pz-p^2x-pqy-qz+pqx+q^2y=1$

$z(p-q)=1+p^2x+pqy+\cancel{\;}-pqx-q^2y$

$$z=\frac{1+p^2x+pqy-pqx-q^2y}{p-q}$$

$$z=\frac{1+(p-q)(px+qy)}{p-q}$$

$$z=\frac{1}{p-q}+px+qy$$

$\Rightarrow z=\frac{1}{p-q}+px+qy$

$$z=\frac{1}{a-b}+ax+by$$

Note:- $f(z,p,q)=0$

$z=g(t)$ $t=x+ay$

$p=\frac{dz}{dt}$

$q=a\frac{dz}{dt}$

} replace in $f(z,p,q)$

solution

$\phi(z,t,b)=0$

solution $\phi(z,x+ay,b)=0$

$f(p,x)=g(q,y)=K$

$p=u(K,x)$

$q=v(K,y)$

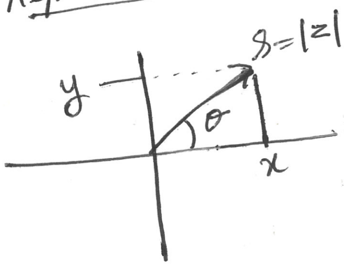
$dz=p\,dx+q\,dy$

$z=\int p\,dx+\int q\,dy+c$

Complex Variables

Complex Numbers: A number of the form $x+iy$ is called a complex number where x and y are real numbers and $i=\sqrt{-1}$.

Representation of a complex number in the complex plain.



$\theta=\tan^{-1}\left(\frac{y}{x}\right)$

Congujate of a complex number

Congujate of $z=x+iy$ is denoted by $\bar{z}$ and is given by

$\bar{z}=x-iy$

Modulus of a complex number

Modulus of $z=x+iy$ is a positive real number and is given by $|z|=\sqrt{x^2+y^2}$.

Polar form of a complex number.

$$z=x+iy = |z|e^{i\theta} = re^{i\theta}$$

$$\boxed{\begin{aligned} r&=|z|=\sqrt{x^2+y^2} \\ \theta&=\tan^{-1} y/x \\ x&=r\cos\theta \\ y&=r\sin\theta \end{aligned}}$$

r is called modulus of the complex no. and θ is called amplitude or argument of the complex no.

Complex Function

→ Corresponding to each point z in the region R of z-plane there corresponds a unique image W in the region R' of W-plane then W is called a complex function

$$\boxed{W=f(z)=U(x,y)+iV(x,y)}$$

$z=x+iy$

u,v are functions of x,y.

Neighbourhood of a point

$N_\delta(z_0)=\{z : |z-z_0|<\delta\}$

Set of all complex no.s which lies inside the circle with center as z_0 and radius as δ is called Delta Neighbourhood $[N_\delta(z_0)]$ of the point z_0.

$\Rightarrow f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ irrespective of the path $\delta z \to 0$

→ If this limit exist then the function $f(z)$ is said to be differentiable at point z. If differentiable the C.R @ that point.

→ Let $w = f(z) = u + iv$ is differentiable then we have the following

$$\boxed{\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

$$\boxed{\frac{dw}{dz} = f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$ C.R equation

Analytic function is also called Holomorphic function

Analyticity Definitions

→ **Analytic at a point** – A function $f(z)$ is said to be analytic at a point if $f'(z)$ exists not only at the point but also in some immediate neighbourhood of the same.

→ A function $f(z)$ is said to be analytic in the region if $f'(z)$ exists at every point on the region.

→ **Entire function**: $f(z)$ is said to be an entire function if it is analytic throughout the finite complex plane.

→ **Note I**: The complex functions $\boxed{f(z) = \sin z, \cos z, e^z, \sinh(z)}$, $\cosh(z)$ and every polynomial in z (ie $f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$) are entire functions.

→ **Note II**: If $f(z)$ and $g(z)$ are two analytic functions then

1. $a f(z) \pm b g(z)$
2. $\frac{f(z)}{g(z)}$ provided $g(z) \neq 0$
3. $f(z) \cdot g(z)$

are also analytic functions.

Singularity

A point at which the function fails to be analytic is called singularity.

eg:) $f(z) = \sqrt{z} \Rightarrow f'(z) = \frac{1}{2\sqrt{z}}$ $z=0$ singularity

For Analyticity C·R must be satisfied at all points & also U_x, V_x, V_y, V_y must exist & continous at all points.

The complex function $f(z) = u+iv$ is analytic in the region if and only if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ exist in the region, and

$\boxed{U_x = V_y}$ and $\boxed{U_y = -V_x}$ (ie C·R equations)

C·R equations in polar form

$f(z) = u(r,\theta) + iv(r,\theta)$ be the complex function in polar form

then the C·R equations are

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \qquad \boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

$u_r = \frac{1}{r} V_\theta$

$u_\theta = -r V_r$

Note ★★★★

if $F(z) = U+iV$ and $U-V$ is given

then $(1+i)F(z) = U-V+i(U+V)$

$G(z) = R(x,y) + iQ(x,y)$

Construction of analytic functions (Holomorphic)

Let $f(z) = U(x,y) + iV(x,y)$ is Analytic function.

then $f(z) = U(z,0) + i(V(z,0))$

I) When real part is given.

$$f(z) = \int \frac{\partial u}{\partial x}\Big|_{(z,0)} dz - i\int \frac{\partial u}{\partial y}\Big|(z,0)dz + c$$

Always do not forget the i when integrating

$$V = -\int_{y\text{ constant}} \frac{\partial u}{\partial y} dx + \int \left(\text{terms of } '\frac{\partial u}{\partial x}' \text{ without } 'x'\right) dy + c$$

II) If imaginary part is given

$$f(z) = \int \frac{\partial v}{\partial y}\Big|_{(z,0)} dz + i\int \frac{\partial v}{\partial x}\Big|_{(z,0)} dz + c$$

$$u = \int_{y\text{ constant}} \frac{\partial V}{\partial y} dx - \int \left(\text{terms of } '\left(\frac{\partial v}{\partial x}\right)' \text{ without } 'x'\right) dy + c$$

Real part — $U(x,y) \rightarrow$ potential function

Imaginary part — $V(x,y) \rightarrow$ stream function

Harmonic functions

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0}$$

A function which satisfy Laplace equation is called harmonic functions.

Note: Let $f(z) = u(x,y) + iv(x,y)$ is an analytic functions, then u & v are harmonic functions. ie $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

and u & v are called harmonic congugate of each other.

Note: Any two harmonic functions need not be analytic (may not obey C.R. eq).

Note: If $f(z) = u(x,y) + iv(x,y)$ is an analytic function, then $u(x,y) = c$ & $v(x,y) = c$ are orthogonal trajectories of each other (ie the family of curves always intersect at 90°.)

---

1) Given a complex no. z, then $ze^{i\alpha}$ (α = real numbers) amounts to?

Solution
$z = re^{i\theta} \cdot e^{i\alpha} = re^{i(\theta+\alpha)}$

rotation anticlockwise

2) $\left|\frac{z-3}{z+3}\right| < 2 \Rightarrow \frac{(x-3)^2 + y^2}{(x+3)^2 + y^2} < 4 \Rightarrow 3x^2 + 27 + 30x + 3y^2 > 0$

$x^2 + 9 + 10x + y^2 > 0$

$(x+5)^2 + y^2 > 16$ (b)

$\left|\frac{(x-3) + iy}{(x+3) + iy}\right| < 2$

$\frac{x^2 + 9 - 6x + y^2}{x^2 + 9 + 6x + y^2} < 4$

3) $f(z) = x^2 + iy^3$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 3y^2$ (c)

$\frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0$

Note:- C.R equation is satisfied only at (zero, zero) ∴ differentiable only at $(0,0)$ ∴ Not analytic

4) $f(z) = u + iv \Rightarrow f'(z)$ exist

∴ $f'(z) = \frac{df}{dz} = \frac{\partial f}{\partial x}\frac{dx}{dz} + \frac{\partial f}{\partial y}\frac{dy}{dz}$

$= \frac{\partial w}{\partial x} 1 + \frac{\partial w}{\partial y} \cdot -i$

$z = x + iy$

$dz = dx$

Note: if CR equation is only satisfied at a point $\Rightarrow$ ~~only~~ differentiable at that point but not analytic

5) If $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$= 3x^2 - 3y^2 - i(-6xy)$

(a) $= 3x^2 - 3y^2 + i6xy$

6) $f(z) = z\,\mathrm{Im}(z)$

~~$f(z) = \ldots\mathrm{Im}(z)$~~

$f(z) = \cancel{x+iy}\ (x+iy)y$

$f(z) = xy + iy^2$

$\frac{\partial u}{\partial x} = y \quad \frac{\partial v}{\partial y} = 2y$ only at $(0,0)$

$\frac{\partial u}{\partial y} = x \quad \frac{\partial v}{\partial x} = \cancel{2y}\ 0$ (a)

08) $f(z) = \int \frac{\partial u}{\partial x}\Big|_{(z,0)} dz - i\int \frac{\partial u}{\partial y}\Big|_{(z,0)} dz + C$

$f(z) = \int A\, dz - i\left[\int B\, dz + C\right]$

$f(z) = Az - iBz + C$ (a)

09) $\frac{\partial v}{\partial y} = -6xy$

$-\frac{\partial v}{\partial x} = -(3x^2 - 3y^2)$

$= 3y^2 - 3x^2$

$u = -3x^2y + y^3$ ∴ (d)

* $f_1(z) = z^2$ & $f_2(z) = \bar{z}$ be two complex functions & $\bar{z}$ is complex conjugate of z then we have the following? ______.

f_1 analytic f_2 not analytic.

* $f(z) = x^2 + ay^2 + ibxy$ is analytic then find a, b.

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = bx \Rightarrow b = 2$

$\frac{\partial u}{\partial y} = 2ay = -\frac{\partial v}{\partial x} = -(by) = -2y \Rightarrow a = -1$

∴ $a = -1,\ b = 2$

* $u = 2xy$ find v if function is analytic

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2y$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2x$

$v = \int -2x\, dx + \int 2y\, dy + C$

$v = -\cancel{2}x^2 + y^2 + C$

$v = y^2 - x^2 + C$

10) $u = \log r$ then $v = ?$ Function is analytic.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} = 1 \qquad \Rightarrow \underline{v = \theta + c} \quad (c)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) = -\frac{1}{r}\,0 = 0$$

---

11) Real part of $f'(z) = 2(x+1)$ $\quad f(0) = 2$, $f(i) = 1+2i$ then $\mathrm{Im}(f'(z)) =$

$$\frac{\partial u}{\partial x} = 2 = \frac{\partial v}{\partial y} \qquad \Rightarrow \underline{v = 2y} \quad (c)$$

$$\frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}$$

Note:- An Analytic function is infinitely differentiable in other words. if $f(z)$ is analytic all higher order derivatives of $f(z)$ exists and are also analytic.

---

12) $u = e^y \sin x$ real part of analytic $f(z)$ then $f(z) = ?$

$u_x = e^y \cos x$

$u_y = e^y \sin x$

$$f(z) = \int e^0 \cos z\, dz - i\int e^0 \sin z\, dz$$

$$f(z) = \sin z + i\cos z + c = i(\cos z - i\sin z) + c$$

$$\underline{i\,e^{-iz} + c}$$

~~$f(z) = i\sinh\left(\frac{z}{i}\right) + \cosh z + c$~~

~~$= i\sinh(iz) + \cosh z + c$~~

~~$= \cosh(z) - i\sinh(iz) + c$~~

~~$= \frac{e^z + e^{-z}}{2} - i\left(\frac{e^{iz} - e^{-iz}}{2}\right) + c$~~

~~$= \frac{1}{2}[$~~

$i\sin\theta = \sinh(i\theta)$

$i\sinh(\theta) = \sin(i\theta)$

$\cos i\theta = \cosh\theta$

$\cosh i\theta = \cos\theta$

When a harmonic function is given and asked to find the other congugate harmonic it can be taken as either real or imaginary part of the analyic function until explicitly given. The two result would be negative of each other.

---

14) Analytic $f(z)$ $\quad v = e^x(y\cos y + x\sin y)$

$$v_x = e^x(y\cos y + x\sin y) + \sin y\, e^x$$

$$v_y = e^x(\cos y - y\sin y + x\cos y)$$

$$+i\int v_{x(z,0)}\, dz + \int v_{y(z,0)}\, dz$$

$$= \int e^z(1+z)\, dz + i\int e^z\, 0\, dz = (1+z)e^z - 1e^z + c$$

$$= (1+z)e^z - e^z + c$$

$$= ze^z + c$$

15) $f(z) = (u+iv)$ $\quad u = 2x(1-y)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d(2(1-y))}{dx} = 0$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{d(-2x)}{dy} = 0$$

~~S1 = true~~.

$$\frac{\partial u}{\partial x} = 2(1-y)$$

$$\frac{\partial u}{\partial y} = -2x$$

$$V = \int 2x\,dx + \int 2(1-y)\,dy$$

$$V = x^2 + 2\left(y - \frac{y^2}{2}\right)$$

$$V = x^2 - y^2 + 2y + c$$

$\therefore$ S2 true

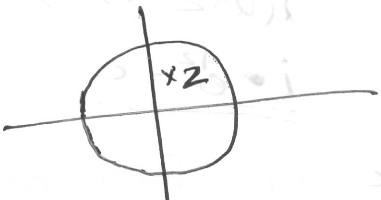
$$f(z) = 2z + iz^2$$

S3 = true

$$\frac{\partial^2 f}{\partial x \partial y} = 4\left(\frac{\partial f}{\partial z}\right)$$

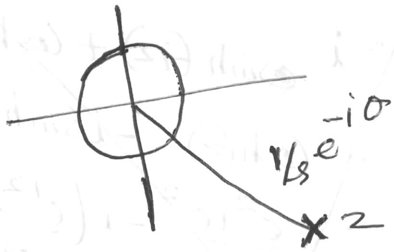
(d)

*) ~~Note~~ ~~Real and Imagin~~ A point z has been plotted in the complex plane as shown in the figure. Then the plot of the complex no. $y = \frac{1}{2}$ is.



$$y = \frac{1}{z} = \frac{1}{8}e^{-i\theta}$$

Solution



* Complex integration.

Complex line Integral = integration along a continuous curve C

$$\boxed{\int_C f(z)\,dz}$$

16) If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1,1) & (2,3) then Evaluate $\oint (12z^2 - 4iz)\,dz = $ ______

$$= \left[4z^3 - 2iz^2\right]_{(1,1)}^{(2,3)} = 4\left[4(2+3i)^3 - 2i(2+3i)^2\right] - 4\left[(1+i)^3 - 2i(1+i)^2\right]$$

$$= \cancel{4(1+i)^3 - 2i(} \quad = \left[4(\underbrace{4-9}_{-5}+12i)(2+3i) - 2i(\underbrace{4-9}_{-5}+12i)\right] - \left[4(\underbrace{1-1}_{0}+2i)(1+i) - 2i(\underbrace{1-1}_{0}+2i)\right]$$

$$= \left[4(-10-15i+24i-36) + 10i + 24\right] - \left[8i - 8 + 4\right]$$

$$= \left[4(-46+9i) + 10i + 24\right] - (8i-4) = \left[-184 + 36i + 10i + 24 - 8i + 4\right]$$

$$\underline{\underline{38i - 156}}$$

$$=$$

Find $\oint (x^2+iy^2)\,dz$ where $C(x=y)_0^1$

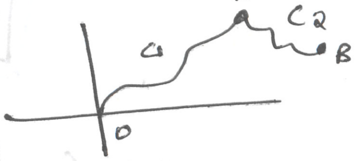
$$= \int_0^1 (x^2+ix^2)\,dx(1+i) = \int_0^1 \cancel{x^2+ix^2}\, x^2(1-1+2i)\,dx$$

$$= 2i\left[\frac{x^3}{3}\right]_0^1 = \underline{\underline{\frac{2i}{3}}}$$

~~If f(z)~~

<u>Note:</u> If the curve C is obtained by combining C_1 & C_2 then

$$\oint F(z)\,dz = \int_{O}^{A}{}_{C_1} F(z)\,dz + \int_{A}^{B}{}_{C_2} F(z)\,dz$$



~~eg:~~

Note :- If $F(z)$ is analytic or entire function then integration can be done in dz itself treating z as variable. This integration is path independent.

Cauchy's Integral theorm

$F(z)$ is a analytic function within and on a simple closed curve C then $\boxed{\oint F(z)dz = 0}$

If a function is analytic & ~~cannot~~ residue cannot be found out even by expansion or by formula then simply integrate taking $z = x+iy$ or $z = re^{i\theta}$

eg. $\int\limits_{|z|=1} \log z \, dz = \frac{-2\pi}{i}$

Cauchy's Integral formula

$F(z) = \frac{f(z)}{(z-a)}$ be a complex function such that $f(z)$ is analytic within and on a simple closed curve C. and the point a lies inside the curve C. then

$$\boxed{\oint F(z)\,dz = \oint \frac{f(z)}{(z-a)}dz = 2\pi i f(a)}$$

Derivative of cauchy's integral formula.

Let $f(z)$ is an analytic function within & on a simple closed curve C. & the point 'a' lies inside the curve C. then.

$$\boxed{\int \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i f^{n-1}(a)}{(n-1)!}}$$

eg-1) Let the curve C be rectangle OABC with vertices $O(0,0)$ $A(2,0)$ $B(2,1)$ $C(0,1)$ Then solve the following.

i) $\int z^2 dz = 0$

ii) $\int \frac{z^2}{z^2+4}dz = 0$

iii) $\int \frac{z^2}{z-1/2-i/2}dz = 2\pi i\left[\frac{1}{4} - \frac{1}{4} + \frac{2i}{4}\right]$

$= -\pi$

iv) $\frac{z^2}{(z-1/2-i/2)^3}dz = \frac{2\pi i}{2!}(2) = 2\pi i$

Note: * $|z-z_0|<r$ represents, set of all the points z which lies inside the circle with centre as z_0 & radius as r.

* $|z-z_0|>r$ represents, set of all the points z which lies outside the circle with centre as z_0 & radius as r.

Q17) If $c = |z| = 2$ then evaluate $\int \left(\frac{z^2-1}{z^3-z^2+9z-9}\right) dz$

$$\oint \frac{(z^2-1)}{(z-1)(z^2+9)} dz \qquad \oint \frac{(z^2-1)\, dz}{(z-1)(z+3i)(z-3i)}$$

$$= \int \frac{\frac{(z^2-1)}{(z^2+9)}}{(z-1)} dz = 2\pi i(0) = \underline{\underline{0}}$$

Q 18.) $\int \frac{(z-1)}{(z^2+1)} dz \qquad |z+i| = 1$

$$\oint \frac{\frac{(z-1)}{(z-i)}}{(z+i)} dz = 2\pi i\left(\frac{-i-1}{-2i}\right) = \underline{\underline{+\pi(1+i)}}$$

19.) $\int \frac{e^{2z}}{(z+1)^4} dz \qquad c = |z| = 2$

$$= \frac{2\pi i}{6}\, 8e^{-2} = \underline{\underline{\frac{8\pi i}{3e^2}}}$$

20.) $\int \frac{e^z}{(z^2+\pi^2)^2} dz \qquad |z| = 4$

$$\int \frac{e^z}{(z+i\pi)^2(z-i\pi)^2} dz = \cancel{\int \frac{e^z}{2i\pi}\left(\frac{-1}{(z+i\pi)^2} + \frac{1}{(z-i\pi)^2}\right) dz}$$

Note :- The conventional direction of contour integration is counter clockwise

→ clockwise integration negative must be applied

→ In questions with figure, the directionality must be observed.

Q21) $\int_C \frac{e^{3z}}{(z-\pi i)}\,dz \qquad |z-2|+|z+2|=6$

$\sqrt{4+\pi^2} + \sqrt{4+\pi^2} = 7.44$

$\int = 0$ because outside.

---

* Find $\oint \frac{\cos 2\pi z}{(2z-1)(z-3)} \qquad |z|=1$

$z = 1/2$

$z = 3$

$= 2\pi i \frac{\cos \pi}{2\times\frac{-5}{2}}$

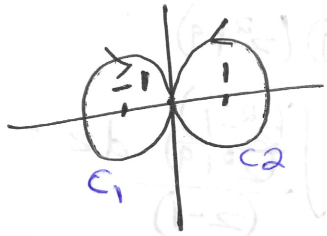
$= \frac{-4\pi i}{2\times 5} = \frac{2\pi i}{5}$

---

* The contour C given below on the complex plane $z = x+jy$

$j = \sqrt{-1}$

Then find $\frac{1}{\pi j}\oint \frac{dz}{z^2-1}$



C1 clockwise
C2 counter clockwise

$$\oint \frac{dz}{z^2-1} = \underset{\text{clock}}{\oint_1 \frac{dz}{z^2-1}} + \underset{\text{counter}}{\oint_2 \frac{dz}{z^2-1}}$$

$$= -\underset{\text{counter}}{\oint_1 \frac{dz}{(z+1)(z-1)}} + \underset{\text{counter}}{\oint_2 \frac{dz}{(z+1)(z-1)}} = -2\pi j \frac{1}{-2} + \frac{2\pi j\, 1}{2}$$

$$= \pi j + \pi j = 2\pi j$$

$$\Rightarrow \frac{1}{\pi j}\oint \frac{dz}{z^2-1} = 2$$

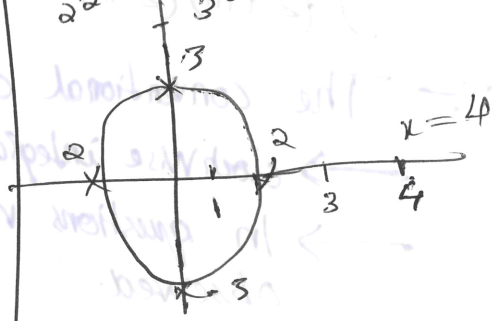
---

* $\oint \left(\frac{4z^2+z+5}{z-4}\right)dz$ where C is

$9x^2+4y^2=36$

$\frac{x^2}{(6/3)^2}+\frac{y^2}{(6/2)^2}=1$

$\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$



$= 0$ outside

Note:) If there is any confusion regarding point inside or outside directly substitute to find out.

Taylor Series.

Let $f(z)$ is an analytic function inside a circle C with centre a then for any point z inside the circle C $\boxed{f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a)}$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n f^n(a)}{n!}} \quad \cancel{\neq} \quad \cancel{\sum_{n=0}^{\infty} c_n f^n(a)}$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n} \quad \boxed{c_n = \frac{f^n(a)}{n!}} \quad \cancel{c_n = \frac{(z-a)^n}{n!}}$$

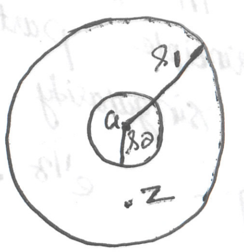
*24) In the Taylor series expansion of $e^z + \sin z$ about $z = \pi$ the coefficient of $(z-\pi)^2$ is.

$$f''(z) = \frac{d}{dz}(e^z + \cos z)$$

$$f''(a) = \frac{e^a - \sin a}{2!} = \frac{e^a - \sin a}{2} = \frac{e^\pi}{2}$$

Laurent's series

Let $f(z)$ is an analytic function in a ring shaped region bounded by 2 concentric circles C_1 & C_2 of radius R_1 & R_2 respectively with centre as 'a'. Then for any point z inside the ring shaped region



$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$
$$+ a_1 (z-a)^{-1} + a_2 (z-a)^{-2} + \cdots$$

$$\Rightarrow f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_n (z-a)^{-n}}_{\text{principle part}} \quad \text{where } a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}} dz$$

~~We call~~.

Removable singularity

In the Laurent's expansion of the function if it contains only the terms of analytical parts then the singularity is called removable singularity. eg. $\frac{\sin z}{z}$ } removable singularity.

Pole of order m

In the laurents expansion of the function, if the principle part contains term until $(z-a)^{-m}$, then $z=a$ is called pole or order m. if $m=1$, then it is called simple pole if $m=2$, then it is called double pole

Note: $\dfrac{f(z)}{(z-a)^k (z-b)^l}$ $z=a$ is a pole of order k if $f(a) \neq 0$
$z=b$ is a pole of order l if $f(b) \neq 0$

$f(a)=0$ or $f(b)=0$ $\Rightarrow$ the numerator has a factor of $\underbrace{(z-a),(z-b)}_{zero}$ respectively and must be taken out

eg:- $\dfrac{\sin z - z}{z^3} = \dfrac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \cdots\right) - z}{z^3}$

removable singularity

eg:- $\dfrac{\sin z - z}{z^2}$ similary is a simple pole
$\sin(0) - 0 = 0 \Rightarrow$ there is a pole @ 0. minimum

Essential singularity.

In the Laurentz expansion of the function if the principle part containes infinite no. of terms than the singularity is called essential singularity.

eg:- $e^{1/z} = 1 + \left(\frac{1}{z}\right) + \dfrac{\left(\frac{1}{z}\right)^2}{2!} + \dfrac{\left(\frac{1}{z}\right)^3}{3!} + \cdots$

Residue The coefficient of $\dfrac{1}{z-a}$ in the laurentz expansion of the function about the point about $z=a$ is called residue of the function at $z=a$.

~~End~~

* Find residue of $e^{1/z}$ @ $z=0$.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2 \times 2} + \frac{1}{z^3 \times 6} + \frac{1}{z^4 \times 24}$$

Residue $= \underline{1}$.

If $z=a$ is a pole of order m

then $\Rightarrow$ $$\boxed{\text{Residue}\,(f(z))\Big|_{z=a} = \frac{1}{(m-1)!} \lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)\right]}$$

if $z=a$ is a simple pole

$\Rightarrow$ $$\boxed{\text{Residue}\,f(z)\Big|_{z=a} = \lim_{z\to a}\left[(z-a)f(z)\right]}$$

30) Sum of the residues of $f(z) = \frac{z-1}{(z+1)(z-3)}$ at its singular point.

$$\text{Res } f(z)\Big|_{z=-1} = \lim_{z\to -1}\left(\frac{z-1}{z-3}\right) = \frac{-2}{-4} = \cancel{-1} = 1/2$$

$$\text{Res } f(z)\Big|_{z=+3} = \lim_{z\to +3}\left(\frac{z-1}{z+1}\right) = \cancel{\frac{4}{2}} = \cancel{2}\ \frac{2}{4} = 1/2$$

$$\Sigma \text{ Re } f(z) = \underline{1}$$

32) $$\text{Re}\left[\frac{1+e^z}{z\cos z + \sin z}\right] \text{ at } z=0$$

$$\lim_{z\to 0}\left(\frac{(z-0)(1+e^z)}{z\cos z+\sin z}\right) = \lim_{z\to 0}\frac{z e^z + (1+e^z)}{\cos z - z\sin z + \cos z}$$

$$= \frac{2}{2} = 1 //$$

31) Sum of Re $f(z) = \frac{\sin z}{z\cos z}$ at $z = \underline{\pi/2}$ & $z = \underline{-\pi/2}$

$$\lim_{z\to \pi/2}\left(\frac{(z-\pi/2)\sin z}{z\cos z}\right) = \lim_{\to \pi/2}\left[\frac{\sin z + (z-\pi/2)\cos z}{-z\sin z + \cos z}\right] = \frac{1}{-\pi/2} = \frac{-2}{\pi}$$

$$\lim_{z\to -\pi/2}\left(\frac{(z+\pi/2)\sin z}{z\cos z}\right) = \lim_{\to -\frac{\pi}{2}}\left[\frac{\sin z + (z+\pi/2)\cos z}{-z\sin z + \cos z}\right] = \frac{-1}{-\pi/2} = \frac{2}{\pi}$$

$$\Sigma \text{Re} = 0 //$$

Q33:

If $f(z) = \frac{2+3\operatorname{cosec} z}{z}$ then Res $f(z)$ at $z=0$

~~$\operatorname{Res} f(z)\Big|_{z\to 0} = \lim_{z\to 0} 2+3\operatorname{cosec} z = 2+3$~~

Note
Take special attention at trignometric, log functions in numerator that may include zeroes.

double pole in hiding

$f(z) = \frac{2}{z} + \frac{3}{z\sin z}$

$\operatorname{Re} f(z) = \operatorname{Re}\left(\frac{2}{z}\right) + \operatorname{Re}\left(\frac{3}{z\sin z}\right)$

$= 2 + \lim_{z\to 0}\left[\frac{d}{dz}\left(\frac{3z}{\sin z}\right)\right] = \cancel{\lim_{z\to 0}\left(\frac{3}{\cos z}\right) = 3}$

$= 2 + \lim_{z\to 0}\left(\frac{\sin z \cdot 3 - 3z\cos z}{\sin^2 z}\right)$

$= 2 + \lim_{z\to 0} \frac{3\cos z - 3\cos z + 3z\sin z}{2\sin z \cdot \cos z}$

$= 2 + \left[\lim_{z\to 0} \frac{3}{2}\frac{z}{\cos z}\right]$

$= 2 + 0 = 2$

Cauchy's Residue Theorm

Let $f(z)$ be an analytic function, within and on a simple closed curve c, except at some finite ~~no. of point not~~ no. of points. poles inside curve c

then $\oint f(z)\,dz = 2\pi i \sum_{n=1} \operatorname{Re}\left[f(z)\right]_{z=P_n}$ When P_1, P_2, P_3 are poles inside curve c.

Q.35) The value of $\oint \frac{e^{1/z}}{z}\,dz$ when $C = |z| = 2$

$$\frac{e^{1/z}}{z} = \frac{1}{z} + \frac{1}{z^2} + \frac{z^{-3}}{2!} + \frac{z^{-4}}{3!} + \cdots$$

$\therefore$ Residue $= 1 \Rightarrow \oint \frac{e^{1/z}}{z}\,dz = \underline{\underline{2\pi i}}$

22) $\frac{1}{2\pi i}\oint \frac{e^{zt}}{(z^2+1)}\,dz \Rightarrow f(z) = \frac{e^{zt}}{(z+i)(z-i)}$

$|z| = 2$

$\Rightarrow \operatorname{Re} f(z)\Big|_{z=i}$

$$\operatorname*{Lt}_{z\to i} \frac{e^{zt}}{(z+i)} = \left(\frac{e^{it}}{2i}\right)$$

$\operatorname{Re} f(z)\Big|_{z=-i}$

$$= \frac{e^{zt}}{(z-i)} \quad \frac{e^{-it}}{-2i}$$

$$\sum \operatorname{Re} = \frac{\cancel{2\pi i}}{\cancel{2i}}\left[e^{it} - e^{-it}\right] = \underline{\underline{2\pi i \sin t}}$$

$$\frac{1}{2\pi i}\oint \frac{e^{zt}}{(z^2+1)}\,dz = \underline{\underline{\sin t}}$$

23) $\oint \frac{\sinh z}{z^4}\,dz \qquad c : |z| = 2$

$$= \oint \frac{\sinh z}{z^4}\,dz = 2\pi i\left[\operatorname{Res}(z=0)\right]$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$= z^{-3} + \frac{1}{z\cdot 3!} + \frac{1}{z^2 5!} + \cdots$$

$$\operatorname{Re} = \left(\frac{1}{6}\right) = \underline{\underline{\frac{\pi i}{3}}}$$

34) $\oint e^z \tan z\,dz \qquad |z| = 2$

$$\oint \frac{e^z \sin z}{\cos z}\,dz \qquad \left(\pi/2, -\pi/2\right) \text{ inside } |z| = 2$$

$$= 2\pi i\left[\operatorname{Re} \text{ at } \pi/2 + \operatorname{Re} \text{ at } -\pi/2\right]$$

$$\operatorname{Re}\ \pi/2 = \operatorname*{Lt}_{z\to \pi/2}\left[\frac{(z-\pi/2)\, e^z \sin z}{\cos z}\right]$$

$$\operatorname*{Lt}_{z\to\pi/2} \frac{(z-\pi/2)e^z \sin z + e^z \sin z + (z-\pi/2)e^z \cos z}{-\sin z}$$

$$= \frac{e^{\pi/2}}{-1} = \underline{\underline{-e^{\pi/2}}}$$

$$\operatorname*{Lt}_{z\to -\pi/2}\left[\frac{(z+\pi/2)\, e^z \sin z}{\cos z}\right]$$

$$= \frac{(z+\pi/2)e^z \sin z + e^z \sin z + (z+\pi/2)e^z \cos z}{-\sin z}$$

$$= \underline{\underline{-e^{-\pi/2}}}$$

$$= -2\pi i\left(e^{\pi/2} + e^{-\pi/2}\right)$$

$$= \underline{\underline{-4\pi i \cosh(\pi/2)}}$$

20) $\oint \frac{e^z}{(z^2+\pi^2)^2} dz = \oint \frac{e^z}{(z+i\pi)^2 (z-i\pi)^2} dz$

$\lim_{z \to i\pi} \frac{d}{dz}\left(\frac{e^z}{(z+i\pi)^2}\right) = \lim_{z \to i\pi} \left(\frac{(z+i\pi)^2 \cdot e^z - e^z(2(z+i\pi))}{(z+i\pi)^4}\right)$

$= \frac{(2i\pi)^2 \cdot e^{i\pi} - [e^{i\pi} 2(2i\pi)]}{(2i\pi)^4} = \frac{+4\pi^2 + 4i}{16\pi^4}$

$= \frac{\pi + i}{4\pi^3}$

$\lim_{z \to (-i\pi)} \left(\frac{d}{dz}\left(\frac{e^z}{(z-i\pi)^2}\right)\right) = \frac{(-2i\pi)^2 e^{-i\pi} - e^{-i\pi}(2(-2i\pi))}{(-2i\pi)^4}$

$= \frac{4\pi^2 - 4i}{16\pi^4} = \frac{\pi - i}{4\pi^3}$

$\Rightarrow \oint \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i \left(\frac{2\pi}{4\pi^3}\right)$

$= \frac{i}{\pi}$

---

25. The coefficient of $1/z$ in the Laurent series expansion $\frac{\log z}{(z-1)}$ valid for $|z|>1$.

$\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right]$

$\log\left(1-\frac{1}{z}\right)^{-1}$ but $\left(1 > \frac{1}{|z|}\right)$

$= (-1)\log\left(1-\frac{1}{z}\right)$

$= -\log(1-z^{-1})$

for $|\frac{1}{z}|<1$

$-\log(1-z^{-1}) = \left[z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \frac{z^{-4}}{4} + \dots\right]$

$\therefore$ coefficient $= 1$

---

27) If $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ then Residue of $f(z)$ @ $z=-2$

Let $(z+2) = u \Rightarrow z = (u-2)$

$\left.\begin{array}{l}(u-2-3)\sin\left(\frac{1}{u}\right) \\ \Rightarrow (u-5)\sin\left(\frac{1}{u}\right)\end{array}\right\} = (u-5)\left[\frac{1}{u} - \frac{1}{u^3 3!} + \frac{1}{u^5 5!} - \frac{1}{u^7 7!}\right]$

$= \left[1 - \frac{1}{u^2 3!} + \frac{1}{u^4 5!} - \frac{1}{u^6 7!} + \dots\right] - \frac{5}{u} + \frac{5}{u^3 3!} - \frac{5}{u^5 5!} + \dots$

Residue $= -5$

26 ✱ $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z=1$

$$\frac{e^{2u+2}}{u^3} = \frac{e^2 e^{2u}}{u^3} = \frac{e^2}{u^3} * \left[1 + 2u + \frac{4u^2}{2!} + \frac{8u^3}{3!} + \cdots\right]$$

$$= \left(\frac{e^2}{u^3} + \frac{2e^2}{u^2} + \frac{2e^2}{u} + \frac{4e^2}{3}\right)$$

constant $= \dfrac{8e^2}{3!} = \dfrac{4e^2}{3}$

residue $= 2e^2$

---

28 ✱ $f(z) = \dfrac{z}{(z+1)(z+2)}$ about $(z=-2)$ $(z+2)=u$

$$f(z) = \frac{u-2}{(u-1)u} = \frac{-(u-2)}{(1-u)u}$$

$$= -\frac{(u-2)}{u}\left[1+u+u^2+u^3+u^5\right] \quad \text{for } |u|<1 \text{ ROC}$$

$$= \frac{2}{u} - 1\left[1+u+u^2+u^3+u^5\right) = \frac{2}{u} + 1 + u + u^2 + u^4$$

If $\sum a_n (z-z_0)^n$ is a power series

then $\displaystyle r = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$ or $\displaystyle r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

gives the radius of convergence

$|z-z_0| = r$ = circle of convergence

$|z-z_0| < r$ = region of convergence

$e^z = 1 + \dfrac{z^1}{1!} + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!} + \cdots$

$\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots$

$\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots$

$\sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \dfrac{z^7}{7!} + \cdots$

$\cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \dfrac{z^6}{6!} + \cdots$

$\log(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \dfrac{z^5}{5} + \cdots$ $|z|<1$ ROC

$\log(1-z) = -\left(z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \dfrac{z^4}{4} + \dfrac{z^5}{5} + \cdots\right)$ $|z|<1$ ROC

$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$ $|z|<1$

$(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$ $|z|<1$

$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$ $|z|<1$

$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3$ $|z|<1$

Converting expressions into Laurent's series

Type I) e^{z} at point z_0 put $(z-z_0)=u$ then $z=u+z_0$

$\Rightarrow e^{z}=e^{u+z_0}=e^{z_0}\cdot e^{u}$

Type II $\dfrac{1}{(au+b)} = \dfrac{1}{a(u+\frac{b}{a})} = \dfrac{1}{b(\frac{au}{b}+1)} = \dfrac{(1+\frac{au}{b})^{-1}}{b}$ $\boxed{\begin{matrix}\left|\frac{au}{b}\right|<1\\ |u|<\left|\frac{b}{a}\right|\end{matrix}}$

Type III $\dfrac{1}{(au+b)} = \dfrac{1}{au(1+\frac{b}{au})} = \dfrac{1}{au}\left(1+\frac{b}{au}\right)^{-1}$ $\left|\frac{b}{au}\right|<1$

$|u|>\frac{b}{a}$

IV $\log[f(z)]$ convert to form $\log(1-z)$ or $\log(1+z)$ to apply log maclaurin series.

Zeros of $\sin z =$ $z=n\pi$ $n=0,\pm1,\pm2$

Zeros of $\cos z =$ $z=(2n+1)\pi/2$ $n=0,\pm1,\pm2$

Zeros of $\sinh z =$ $z=n\pi i$ $n=0,\pm1,\pm2$

Zeros of $\cosh z =$ $(2n+1)\pi/2\, i$ $n=0,\pm1,\pm2$

$\rightarrow |z|^2 = z\bar{z}$

$\rightarrow z^n = r^n e^{i\theta n}$ This can be used to find root of complex no.

n^{th} root of $z = z^{1/n} = r^{1/n} e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}$ $k=0, n-1$

eg:- $i^i = (e^{i\pi/2})^i = e^{-\pi/2}$

principle argument $\pi > \theta - \pi$

$\rightarrow \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \rightarrow |z_1 z_2| = |z_1|\cdot|z_2|$

← (★) See left page

★ Type II & III technique can also be used to convert to form ~~1+ (…)~~ $(1+u^n)^{-1}$ or $(1+u^n)^{-2}$ or $(1+u^n)^{-2}$ or $(1+u^n)^{2}$

Type IV | for | $\sin z$ ~~apply~~ about z_0

put $z - z_0 = u$

then $z = u + z_0$

$\sin(u+z_0) = \cancel{\sin u \,(\sin z_0)}$.

$\sin u(\cos z_0) + \cos u(\sin z_0)$

or use shifting if $z_0 = n\pi, n\pi/2$

⟹ Similarly apply identity for $\cos z$ as well

To find the order of pole(z_0) - if numerator is zero. ~~then~~ then derivative the denominator n times to get non zero value at singularity $(z_0) \Rightarrow n$ is order

To find the order of pole (z_0) if both numerator & denominator is zero. Try to cancel the zero & pole by factorising or expansion

PROBABILITY & STATICS

4 marks

Basic Probablity
Random Variable
Probablity distribution
Correlation & Regression

→ Random experiment :- Unpredictable outcome of the experiment is known as a random experiment example:- Tossing a unbiased coin.

eg: → Rolling a die
eg: Drawing a card from pack of 52.

→ Sample space :- The collection of all possible outcomes of the random experiment is known as a sample space and it is denoted by S.

→ Event : The outcome of the experiment is known as event. event is the subset of the sample space.

→ Definition of probablity: Probablity of even is defined as the ratio between the favourable cases of the event and the total no of ways in the experiment. (The outcomes ~~must be~~ mutually exclusives, equally likely, & exhaustive cases) Therefore $P(e) = \frac{m}{n}$ $m < n$

Rules (Axiomatic approach)

1) $P(S) = 1$
2) $0 \leq P(E) \leq 1$
3) $P(\phi) = 0$

if $P\left(\bigcup_{i=1}^{m} E_i\right) = P(S) = 1$

E_1, E_2 etc are called mutually exhaustive.

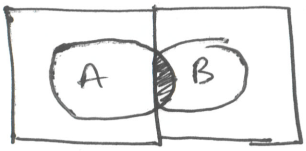
ie) $E_1 \cup E_2 \cup E_3$ etc $= S$

$\left\{\bigcup_{i=1}^{m} E_i\right\} = \{S\}$

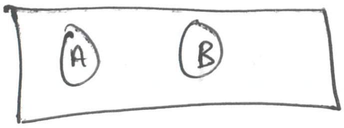
4) if $\boxed{P\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} P(E_i)}$

then the set of events E_i are said to be mutually ~~exhaustive~~ exclusive.

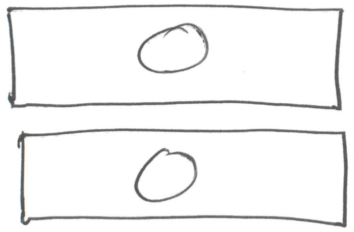
ie) If probablity of union of E_1, E_2, etc is equal to the sum of probablity of E_1, E_2 etc then E_1, E_2 etc are called mutually ~~exclusive~~ exclusive



Mutually dependent — Occurance of one event is associating with other event in the same trail. they are ~~are~~ known as dependent event



Mutually exclusive — Occurance of one event prevents the occurancy of the other is called mutually exclusive event (in the same trail)



Independent events — A Occurance of one event does not depend upon the occurance of same event in a different trail then they are known as

With replacement ⇒ Independent trials
Without replacement ⇒ dependent events.

if A & B are mutually exclusive event

then, $A \cap B = \phi$

$P(A \cap B) = 0$

if independent

$P(B|A) = P(B)$; $P(A \cap B) = P(A) \cdot P(B)$

Results.

Addition theorm

if A&B are arbitrary events	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
if A&B&C are arbitrary events	$P(A \cup B \cup C) = P(A) + P(B) + P(C) - (P(A \cap B) + P(A \cap C) + P(B \cap C)) + P(A \cap B \cap C)$
if A&B are M. Exclusive	$P(A \cup B) = P(A) + P(B)$
if A&B are M. Independent	$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$
if A&B&C are M. Exclusive	$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Product theorem.

→ if A&B are mutually exclusive $\quad P(A \cap B) = 0$

~~$P(A \cap B) = 0$~~

→ if A&B are M. Dependent $\quad P(A \cap B) = P(A) \cdot P(B|A)$

$P(A \cap B) = P(B) \cdot P(A|B)$

→ if A&B&C are M. dependent $\quad P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

→ if A&B are M. Independent $\quad P(A \cap B) = P(A) \cdot P(B)$

If A & B are two arbitrary events

only A : $P(A\cap B^c) = P(A-B) = P(A-(A\cap B))$
$= \underline{P(A) - P(A\cap B)}$

only B : $P(A^c\cap B) = P(B-A) = P(B-(A\cap B))$
$= \underline{P(B) - P(A\cap B)}$

neither A nor B : $P(A^c\cap B^c) = P(\overline{A\cup B}) = \underline{1 - P(A\cup B)}$
none of A, B :

exactly one : $P(A\Delta B) = P((A-B)\cup(B-A))$ since
only one $= P(A-B) + P(B-A)$ } mutually exclusive
$= P(A) - P(A\cap B) + P(B) - P(A\cap B)$
$= P(A) + P(B) - 2P(A\cap B)$
$= \underline{P(A\cup B) - P(A\cap B)}$

Both A : $\underline{P(A\cap B)}$

either or atleast one : $\underline{P(A\cup B)}$

atmost one $= P(A^c\cap B^c \cup P(A\Delta B))$ } M. Exclu
$= P(A^c\cap B^c) + P(A\Delta B)$
$= 1 - P(A\cup B) + P(A\cup B) - P(A\cap B)$
$= \underline{1 - P(A\cap B)}$

If A & B are Mutually Exha & Mutually Exclusive $\Rightarrow P(A\cup B) = 1$
$P(A\cap B) = 0$

$P(A\cap B^c) = P(A)$
$P(B^c\cap A) = PB$
$P(A^c\cap B^c) = 0$
$P(A\Delta B) = 1$
atmost one $= 1$

If A&B are independent $P(A\cap B)=P(A)P(B)$

$$P(A\cap B^c) = P(A) - P(A)P(B) = P(A)[1-P(B)] = P(A)P(B^c)$$

$$P(A^c\cap B) = P(B) - P(A)P(B) = P(B)(1-PA) = P(A^c)P(B)$$

if A & B are independent then compliment events are also independent

$$P(A^c\cap B^c) = 1-P(A\cup B) = 1-P(A)-P(B)+P(A)P(B)$$
$$= 1-P(A)+P(B)[P(A)-1]$$
$$= [1-P(A)][1-P(B)]$$
$$= P(A^c)\cdot P(B^c)$$

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

$$P(A^c/B) = \frac{P(A^c\cap B)}{P(B)} = \frac{P(B)-P(A\cap B)}{P(B)}$$

$$\Rightarrow \boxed{P(A/B)+P(A^c/B)=1}$$

$$P(A/B^c) = \frac{P(A\cap B^c)}{P(B^c)} = \frac{P(A)-P(A\cap B)}{P(B^c)} = \frac{P(A)-P(A\cap B)}{1-P(B)}$$

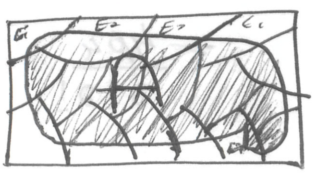
$$P(A^c/B^c) = \frac{P(A^c\cap B^c)}{P(B)^c} = \frac{1-P(A\cup B)}{1-P(B)}$$

$$\Rightarrow \boxed{P(A/B^c)+P(A^c/B^c)=1}$$

Baye's theorm

If E_1, E_2, and so ... E_n are mutually exclusive ~~exhaustive~~ events $[P(E_i)\neq 0]$ which is a subset of $\bigcup_{i=1}^{n} E_i$ such that A is an arbitrary event

then $\boxed{P(A) \text{ is } \sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$



$$A = (E_1\cap A)\cup(E_2\cap A)\cup\ldots$$

$$P(A) = P(E_1\cap A)+P(E_2\cap A)+\ldots$$

$$P(A) = \sum_{i=1}^{n} P(E_i\cap A)$$

$$P(A) = \sum_{i=1}^{n} P(E_i)\cdot P(A/E_i)$$

$$\boxed{P(E_i/A) = \frac{P(E_i \cap A)}{P(A)}} = \frac{P(E_i)\cdot P(A_i/E_i)}{\sum_{n=1}^{n}\left[P(E_i)\cdot P(A/E_i)\right]}$$ (reverse probablity)

Baye's theorm

$$P(E_i/A) = \frac{P(E_i)\cdot P(A|E_i)}{\sum_{n=1}^{n}\left[P(E_i)\cdot P(A|E_i)\right]}$$

Note sum of all reverse probablity of all reverse probablity $=1$

$\sum P(E_i/A) = 1$

Cards

4 suits $\longrightarrow$ Diamond, hearts, clubs, spades

13 numbers 1-13

3 face cards — K, Q, J

atleast | min ($\geq n$)

atmost | max ($\leq n$)

and product ($\cap$)

or sum ($\cup$)

- Baye's theorm is applied when conditional probablity (reverse probablity) of a otherwise known (known without condition) event is asked.

Example

Q If 3 coins are tossed find the probablity of getting atmost one head.

Solution $4/8 = 1/2$

Q Find the probablity that atleast one tail. $= 7/8$ (same data)

Q Find the probablity that atleast one head and atmost one tail

HHH, HHT, HTH, THH $= 4/8$

Find the probablity that atmost 2 head and atmost two tail. when 4 coins are tossed.

HHTT, HTTH, THHT, HTHT, THTH, TTHH

$6/16$

$4c_2 = nc_r = \frac{n!}{(n-r)r!}$

$= \frac{4!}{2!2!}$

Q) Find the probablity that atmost 2 tail and atleast one head

$\frac{4P_4}{4!}$ = 0 T, 4 H = 1 $P = 11/16$ TEXT $\frac{4C_3}{2!}$

1 T, 3 H = 4

$\frac{4P_4}{3!}$ =

2 T, 2 H = 6

$\frac{4P_4}{2!2!}$

Q) Find the probablity that atleast two heat and atleast 2 tail when 6 coins are tossed.

$\frac{6!}{5!\cdot 1!} = 6$

~~1H, 3T = 4~~

~~2H, 2T = 6~~

~~P = 10/16~~ 64

2H 4T = 15

3H 3T = 20 $= 50/64 = \frac{50}{64}$

4H 2T = 15

Q) P(Atmost 2H or atmost 2T) 6 coins.

0H 6T — 1

1H 5T — 6

2H 4T — 15 $\frac{44}{64} = \frac{22}{32}$

4H 2T — 15

5H 1T — 6

6H 0T — 1

Q) N coins are tossed at a time. Find the probablity of getting the head odd no. of times.

Binomial $\sum_{r=1}^{3} nC_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{n-r}$

Favourable $= \frac{n!}{(n-1)!\,1!} + \frac{n!}{(n-3)!\,3!} + \frac{n!}{(n-5)!\,5!} = 2^{n-1}$

$= \frac{2^{n-1}}{2^n} = \frac{1}{2}$

★ sum of odd binomial coefficients = sum of even binomial coefficients $= 2^{n-1}$.

Q) 2 dice are rolled 2 times, find the probablity of getting sum 9.

i) atleast once

ii) exactly once

iii) twice.

iv) atmost once

A = getting a sum of 9 when throwing 2 dice

3,6 — 2

4,5 — 2 $= (4/36) = \frac{1}{9}$

~~atleast once~~ = ~~once~~ ~~NOA~~ twice

$= \cancel{\frac{4}{36}\times\frac{32}{36}} + \frac{4}{36}\times\frac{4}{36}$

i) Atleast once = ONCE + TWICE

$$= \left[\left(\frac{1}{9}\times\frac{8}{9}\right)+\left(\frac{8}{9}\times\frac{1}{9}\right)\right]+\left[\frac{1}{9}\times\frac{1}{9}\right] = \frac{16}{81}+\frac{1}{81} = \frac{17}{81}$$

Independent $P(A\cup B) = P(A)+P(B) - P(A)P(B)$

$$= \frac{1}{9}+\frac{1}{9}-\frac{1}{81} = \frac{18}{81}-\frac{1}{81} = \frac{17}{81}$$

ii) $P(\text{twice}) = P(A\cap B) = \left(\frac{1}{9}\times\frac{1}{9}\right) = \frac{1}{81}$

iii) $P(\text{atmost once}) = 1 - P(A\cap B) = 1-\frac{1}{81} = \frac{80}{81}$

$$= \left(\frac{8}{9}\cdot\frac{8}{9}\right)+\frac{16}{81} = \frac{64+16}{81} = \frac{80}{81}$$

Q) Two dice are rolled find the probablity that (1) neither sum 5 nor sum 6 (2) neither product 6 nor product 9 (3) neither sum 8 nor tie.

Solution 1) $n(S)=36$

$P(5'\cap 6')$

(1,6) — 2	(2,6) — 2	(3,6) — 2	(4,6) — 2	(5,6) — 2
(1,3) — 2	(2,5) — 2	(3,5) — 2	(3,5) — 2	(5,5) — 1
(1,2) — 2	(2,2) — 1	(3,4) — 2	(4,4) — 1	
(1,1) — 1				

or

(1,5) — 2 | (1,4) — 2

(2,4) — 2 | (2,3) — 2

(2,3) — 1

$$1-\frac{9}{36} = \frac{27}{36}$$

2) $P(6'\cap 9') = 1 - P(6\cup 9)$

(3,2) — 2   (3,3) — 1

(6,1) — 2

$$= 1-\frac{5}{36} = \frac{31}{36}$$

b) $P(8'\cap \text{tie}) = 1-P(8\cup \text{tie})$

$$1-\frac{10}{36} = \frac{26}{36}$$

(6,2) — 2   (3,3) 1

(4,4) — 1   (2,2) 1

(5,3) — 2   (1,1) 1

(5,5) 1

(6,6) 1

Q) A card is drawn from the pak of 52 cards. Find the probablity that

1) Neither a diamond nor a king
2) Neither a club nor a red
3) Neither a 10 nor a jack.
4) Neither a face nor a club

1) $P(D' \cap K')$

$= 1 - P(D \cup K)$

$= 1 - \left(\frac{16}{52}\right)$

$= 1 - \frac{16}{52} = \frac{36}{52}$

2) $1 - P(C \cup R)$

$= 1/4$

3) $1 - P(10 \cup J)$

$1 - [P(10) + P(J) - P(10 \cap J)] = 1 - \left[\frac{4+4}{52}\right] = \frac{44}{52}$

4) $1 - [P(F) + P(C) - P(F \cap C)]$ $1 - \left[\frac{12}{52} + \frac{13}{52} - \frac{3}{52}\right]$

$= \frac{30}{52}$

Q) 4 cards are drawn at random from the pack of 52 cards. find the probablity that

1) All four cards are drawn from same suite
2) No two cards are drawn from same suite

a) $p(\text{same suite}) = \cancel{P(A)}\ \frac{4(13C_4)}{52C_4}$ } 4 cards drawn from one suite

b) $P(\text{different suite})\ \frac{13C_1 \cdot 13C_1 \cdot 13C_1 \cdot 13C_1}{52C_4}$

$52C_4$

$= \frac{52!}{48!\,4!}$

$\frac{52 \times 51 \times 50 \times 49}{2 \times 3 \times 4}$

Q) A determinate is selected from the set of all determinants of order 2. with the elements (0 and or 1)

the selected determinat is non zero.

Find the probablity that the ~~seler~~ selected determinat is non zero.

Sol

$\begin{bmatrix} a_{01} & b_{01} \\ c_{01} & d_{01} \end{bmatrix}$

$n(S) = (2 \times 2 \times 2 \times 2) = 2^4 = 16$

$2 + 8 = 10$

$1 - P(D = 0)$

$1 - \frac{10}{16} = \frac{6}{16}$

$$P(\Delta=+1) + P(\Delta=-1) + P(\Delta=0) = 1$$

$\underbrace{P(\Delta=+1) + P(\Delta=-1)}_{\text{M. Exclusive}}$

$$P(\Delta\neq 0) = P(\Delta=1) + P(\Delta=-1)$$

$$= \frac{3}{16} + \frac{3}{16} = \frac{6}{16}$$

$$P(\text{non negative}) = P(\Delta\neq -1)$$

$$= 1-(P\,\Delta=-1)$$

$$= \frac{13}{16}$$

$$P(\text{zero}) = \frac{10}{16}$$

Q) A, B are the 2 players A selecting the card ~~of~~ from the pack of 52. B selecting the no. on the die. A declared as the winner if he gets the diamond card before B getting the composite number. If A starts the game, what are the winning chances of player A, player B.

A winning $\frac{13}{52}\times$

First winning

Second wining

$$\frac{39}{52}\times\frac{4}{6}\times\frac{13}{52}$$

$P(A) = 1/4$ $P(A^c) = \frac{39}{52}$

$P(B) = 1/3$ $P(B^c) = \frac{2}{3}$

$$= \frac{13}{52}\left(\frac{1}{1-\frac{78}{156}}\right)$$

$$= \frac{13}{52}\;\frac{156}{78}$$

$$\frac{26}{52} = 1/2$$

$$\begin{array}{r}156-\\78\\\hline 078\end{array}$$

$$P(\text{Win } B) = P(A^c).P(B) + P(A^c)P(B^c)P(A)^cP(B) + (P(A^c))^2P(B^c)^2P(A)^cP(B) + P(A^c)^3P(B^c)^3P(A)^cP(B)$$

First term $= P(A^c)P(B)$

$r = P(A^c)P(B^c)$

$$\text{infinite sum} = \left(\frac{a_0}{1-r}\right)$$

$$= \frac{39}{52}\times\frac{1}{3}\left(\frac{1}{1-\frac{39}{52}\times\frac{2}{3}}\right)$$

$$P(\text{Win } A) = P(A) + P(A^c)\,P(B^c)\,P(A) + (P(A^c)P(B^c))^2\,P(A)$$

$$P(A)\left[1 + P(A^c)P(B^c) + (P(A^c) + P(B^c))^2 + \dots\right.$$

$$= \frac{1}{4}\left(\frac{1}{1 - P(A^c)P(B^c)}\right) = \frac{2}{4} = \underline{\underline{1/2}}$$

$$P(A\,\text{win}) = \left(\frac{P(A)}{1 - q_A q_b}\right)$$

$$P(B\,\text{win}) = \left(\frac{q_a\,P(B)}{1 - q_B q_a}\right)$$

$P(A)$ = chance of A winning

q_A — A failing.

Fundamental principle of counting

If an event can occur in 'm' different ways following which another event can occur in n different ways, then the total number of occurance of the events in the given order is $m \times n$.

If an event can occur in 'm' different ways following which the same event can also occur in 'n' different ways without a common between the two, then the total number of occurance in the given event is $m+n$.

${}^nP_r$ — Permutation

→ N objects taken r @ a time (arrangement important) $= \frac{n!}{(n-r)!}$

→ No. of permutation of n objects where p objects are of same kind and rest are different is $n!/p!$

→ if P_1 objects are of one kind and P_2 objects are of second kind etc upto n. then No of permutation is $\frac{n!}{p!\,p_2!\dots p_n!}$

Combination (order not important)

→ Selection of N objects taking r at a time = Arrangement of N objects taking r at a time
No. of way r objects can be arranged.

$$\Rightarrow {}^nC_r = \frac{{}^nP_r}{r!} \quad \therefore {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

→ $ {}^nC_r \times r! = {}^nP_r$
means – corresponding to each combination in ${}^nC_r$ there are $r!$ permutation for those r objects because r objects of every combination can be arranged $r!$ no of ways.

properties

1) ${}^nC_r = {}^nC_{n-r}$
2) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
3) No. of ways to select zero or more objects out of n distinct objects $= 2^n$
ie) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots {}^nC_n = 2^n$
4) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{n-1}$
Sum of odd no. of Combinations = Sum of even no. of combinations $= 2^{n-1}$

25/08/2019

Q:- A die is rolled. If the number is odd number, find the probability for prime numbe?

An:- $P(P/O) = \frac{2}{3}$ P - getting Prime number, O - getting odd "

[$P(O) = \frac{3}{6}$; $P(O \cap P) = \frac{2}{6}$ 1, 3, 5 (odd)

$\therefore P(P/O) = 2/3$]

Illy $P(E/P) = \frac{1}{3}$ $\longrightarrow P = 2, 3, 5$

$P(C/E) = 2/3$ $E = 2, \overset{c}{4}, \overset{c}{6}$

Q:- A number is drawn at random from the 50 numbers. Those are {00, 01, 02 ... 49}. Let 'X' denotes the sum of the digits on the number & Y denotes product of the digits on the number. Find the probability that $P(X=3/Y=0)$

An:- $P(X=3/Y=0) = \frac{2}{14} = \frac{1}{7}$

Y=0: 00, 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 20, 30, 40

Q:- A family has two children. If there is a boy, find the probability that other also a boy.

An:- {BB, BG, GB, GG}

$P(B/B) = \frac{1}{3}$

$P(B|1^{st}B) = \frac{1}{2}$

$P(B|G) = \frac{2}{3}$

$P(B|1^{st}G) = \frac{1}{2}$

Q:- 2 dice are rolled if it is a sum 8, find the probability that 1st die has a Prime number?

An:- $P(1^{st} Prime / sum 8) = ?$

Sum 8 = { (3,5)(5,3), (6,2)(2,6), 4,4 }

$P(1^{st} Prime / sum 8) = \frac{3}{5}$

Q:- 60% of the Employees of the company are college graduates. Of these 10% are in the sales department; of these employees who didn't graduate from college are 80% in the sales department. A person is selected at random. Find the probability that the person is in the sales?

An:-

- To find an unknown event from just one known event → Use conditional Probability.
- To find an unknown event from two or more known event → Use total Probability (Provided the known events should be ME)

$P(G) = 60\% = 0.6$ $P(G^c) = 40\% = 0.4$

$P(S/G) = 10\% = 0.1$ $P(S/G^c) = 0.8$

G∩S G^c∩S

G	G^c

$P(S) = P(G \cap S) + P(G^c \cap S)$

$= P(G) \times P(S/G) + P(G^c) \times P(S/G^c)$

$= (0.6 \times 0.1) + (0.8 \times 0.4) = 0.06 + 0.32$

$= 0.38 = 38\%$

[P(S) = 10% of 60% + 80% of 40%]

Q:- These are 2 bags

	blue	Red
bag A	4	2
bag B	3	4

A bag is drawn at random and a marble is taken from it. If it is a red, find the probability that it is in bag B?

An:-

$P(A)=1/2 \quad P(B)=1/2 \quad P(B|R) = ? = \frac{P(B\cap R)}{P(R)}$

Its already known

$\boxed{\text{Use reverse probability} \Rightarrow P \text{ of known event}}$

$P(R/A)=\frac{2}{6} \qquad P(R/B)=\frac{4}{7}$

$P(R)=P(A\cap R)+P(B\cap R)$

$= P(A)\cdot P(R|A) + P(B)\cdot P(R|B)$

$= \left(\frac{1}{2}\times\frac{2}{6}\right)+\left(\frac{1}{2}\times\frac{4}{7}\right) = \frac{1}{6}+\frac{4}{14} = \frac{38}{84}$

$P(B|R) = \frac{P(B\cap R)}{P(R)} = \frac{P(B)\cdot P(R|B)}{P(R)}$

$= \frac{1/2\times 4/7}{\frac{38}{84}} = \frac{24}{38}$

Illy

$P(A|R) = \frac{P(A\cap R)}{P(R)} = \frac{P(A)\cdot P(R/A)}{P(R)} = \frac{\frac{1}{2}\times\frac{2}{6}}{\frac{38}{84}}$

$= \frac{14}{38}$

$\boxed{P(A|R)+P(B|R)=1}$ (bcz ~~numerator~~ denominator is same & ~~denominator~~ numerators are ME)

Q:- There are 4 coins of these 2 are unbiased, one is a head biased and other is a tail biased. A coin is drawn at random & tossed 3 times. It appears head in all the time. Find the probability that head is ~~happened~~ happened in unbiased coin?

An:-
- unbiased – Tail & head (regular coin | fair coin | balanced | true coin)

- Tail biased = both sides are tail
- Head biased = both sides are head

false coin

UB	HB	TB	
2	1	1	=4

$\boxed{H} \quad \boxed{H} \quad \boxed{H}$

1st 2nd 3rd

Independent events

$P(UB|H) = \frac{P(UB\cap H)}{P(H)}$

$P(H/UB) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$

$P(H|HB) = 1\times1\times1 = 1$

$P(H|TB) = 0\times0\times0 = 0$

$P(H) = P(UB\cap H)+P(HB\cap H)+P(TB\cap H)$

$= P(UB)\cdot P(H/UB) + P(HB)\cdot P(H/HB) + P(TB)\cdot P(H/TB)$

$= \left(\frac{2}{4}\times\frac{1}{8}\right)+\left(\frac{1}{4}\times1\right)+\left(\frac{1}{4}\times0\right) = \frac{2}{32}+\frac{1}{4}$

$= \frac{10}{32}$

$P(UB|H) = \frac{P(UB)\cdot P(H|UB)}{P(H)} = \frac{\frac{1}{2}\times\frac{1}{8}}{\frac{10}{32}}$

$= \frac{32/16}{10} = 2/10 = \frac{1}{5}$

Illy

$P(HB|H) = \frac{P(HB)\cdot P(H/HB)}{P(H)} = \frac{4}{5}$

$P(TB|H) = 0$

$\boxed{P(UB|H)+P(HB|H)+P(TB|H)=1}$

Q:- There are 3 machines which has equal capacity for manufacturing the bolt. A bolt is drawn at random & found to be defective. The chance of getting defect bolt from each machine are 1%, 2% & 1%. Find the probability that it is happened in machine B?

An:- $P(B/D) = \frac{P(B \cap D)}{P(D)}$

$P(A) = \frac{1}{3}$ $P(B) = \frac{1}{3}$ $P(C) = \frac{1}{3}$

$P(D) = P(D \cap A) + P(D \cap B) + P(D \cap C)$

$P(D/A) = 0.01$ $P(D/B) = 0.02$ $P(D/C) = 0.01$

$P(D) = P(A) \cdot P(D/A) + P(B) \cdot P(D/B) + P(C) \cdot P(D/C)$

$= (\frac{1}{3} \times 0.01) + (\frac{1}{3} \times 0.02) + (\frac{1}{3} \times 0.01)$

$= \frac{1}{3}(0.01 + 0.02 + 0.01) = \frac{0.04}{3}$

$P(B/D) = \frac{P(B) \cdot P(D/B)}{P(D)} = \frac{\frac{1}{3} \times 0.02}{\left(\frac{0.04}{3}\right)} = \frac{2}{4}$

Illy $P(A/D) = \frac{P(A) \cdot P(D/A)}{P(D)} = \frac{1/3 \times 0.01}{\frac{0.04}{3}} = \frac{1}{4}$

$P(C/D) = \frac{1/3 \times 0.01}{\frac{0.04}{3}} = \frac{1}{4}$

$P(B/D) + P(A/D) + P(C/D) = \frac{2}{4} + \frac{1}{4} + \frac{1}{4} = 1$

P(53 sundays in a leap year)

$= \frac{\text{Extra days}}{\text{no. of days in a week}} = \frac{2}{7}$

$366 \rightarrow 52$ weeks + 2 days

P(53 sundays in non leap year) $= \frac{1}{7}$

$365 \Rightarrow 52$ weeks + 1 day

* $\{1, 2, 3 \cdots 200\}$

number divisible by 6 & 8

$P(\text{div}6 \cap \text{div}8) \Rightarrow \boxed{\text{LCM}(6,8) = 24}$

$= P(\text{div}24) = \frac{8}{200}$

$P(\text{div}6 \cup \text{div}8)$ (or) $= P(\text{div}6) + P(\text{div}8) - P(\text{div}6 \cap \text{div}8)$

$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{1}{4}$

Birthday Prblm

P(none of the person is born in same day)

$= \frac{{}^nP_r}{n^r} = \frac{\text{no. of ways in which all of different}}{\text{total no of ways bdays are alloted.}}$

n = no. of days

r = no. of persons $(r \le n)$

- P(atleast two of them born on same day)

$= 1 - \frac{{}^nP_r}{n^r}$

eg:-

- Group of 15 students

P(atleast 2 of them share same bday in a year) $= 1 - \frac{{}^{365}P_{15}}{(365)^{15}}$

Matching Prblm $\frac{1}{2}-\frac{1}{6}+\frac{1}{4!}-\frac{1}{5!}+\dots=\frac{1}{e}$

- P(none of letter go to right address) $=\frac{1}{e}$

$$=\frac{1}{2.718}=0.36$$

- P(atleast one letter go to right address)

$$=1-P(X=0)$$
$$=1-0.36=0.64$$

P is independent of no. of letters. But Minimum no. of letters = 3

If 2 letters are there then

- P(atleast one letter go to right address) $=\frac{1}{2}$
- P(none of letter go to right address) $=\frac{1}{2}$

Random Variable | Expectation

Random Variable (rv)

connecting the outcomes of the random experiment with real value is known as random variable & its data is known as univariate data.

mathematically, it is represented as

$$\boxed{X: S \Rightarrow R}$$

X — rv; S — Sample space; R — real number

Types of rv

① Discrete rv

② Continuous rv

Random variable

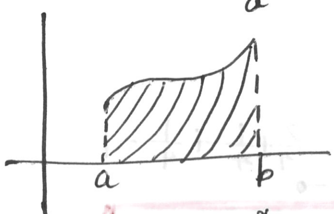
- Integers → Discrete rv → Probability mass function $P(x)$ → Discrete distb^n (BD – Binomial, PD – Poisson)
- Fractions → continuous rv → Probability density function $f(x)$ → Continuous distb^n (ND – normal)

Note:-

- Probability mass func^n (Pmf) measures the probability at integers
- Probability density func^n (PDF) evaluating the probability btw^n the points as an area since it is a piece wise continuous func^n.

$RV \in (a,b)$

$$P(a<x<b)=\int_a^b f(x)dx$$



$$P(x<x_0)=\int_{-\infty}^{x_0} f(x)dx$$

$$P(x>x_0)=\int_{x_0}^{\infty} f(x)dx$$

$$P(x=x_0)=\int_{x_0}^{x_0} f(x)dx=0$$

Expectation (mean)

$$\bar{x} = E(X) = \sum_{x=0}^{n} x \cdot P(x) \quad (x\text{-DRV})$$

$$= \int_{-\infty}^{\infty} x f(x)\,dx \quad (x\text{-CRV})$$

$\sum_{0}^{n} P(x) = 1$ & $\int_{-\infty}^{\infty} f(x)\,dx = 1$ Then only – Expectation exists.

Variance

- $\sqrt{\text{Variance}} = SD$

$$V(x) = E\left[(x - E(x))^2\right]$$

$$\boxed{V(x) = E(x^2) - (E(x))^2}$$

Second order moment

$$V(x) = \sum_{0}^{n} x^2 P(x) - \left(\sum_{0}^{n} x P(x)\right)^2 \quad (x\text{-DRV})$$

$$V(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} x f(x)\,dx\right)^2 \quad (x\text{-CRV})$$

Properties of expectation

If 'x' is rv, and 'a' is constant

① $E(ax) = aE(x)$

② If x & Y are rv

$E(x+Y) = E(X) + E(Y)$

$E(x-Y) = E(X) - E(Y)$

③ If x & y independent rv

$E(X \cdot Y) = E(X) \cdot E(Y)$

④ If $y = ax+b$; a, b constants

$E(Y) = E(ax+b)$

$= E(ax)+b$

$E(Y) = aE(X)+b$

$E(\text{constant}) = \text{constant}$

$E(E(E(x))) = E(x)$ ($\because E(x)$ is a constant)

Properties of Variance

$V(x) = E(x^2) - (E(x))^2$

① $V(ax) = a^2 V(x)$

$[\,V(ax) = E(a^2x^2) - (E(ax))^2$

$= a^2E(x^2) - a^2(E(x))^2$

$= a^2(E(x^2) - (E(x))^2) = a^2V(x)\,]$

Variance is always +ve

$V(-x) = V(x)$

$V(X) \geq 0$ {E(x) can be any values} {but not imaginary}

② if x & Y are independent rv

$V(x+y) = V(x) + V(y)$

$V(x-y) = V(x + (-y)) = V(x) + V(y)$

$$V(x \pm y) = V(x) + V(Y)$$

③ If a, b constant, x & y independent rv.

- $V(ax - by) = a^2V(x) + b^2V(Y)$
- $V\left(\frac{x}{a} - \frac{y}{b}\right) = \frac{1}{a^2}V(x) + \frac{1}{b^2}V(Y)$

If $y = ax+b$; a, b constant

$V(y) = a^2V(x) + V(b)$

$= a^2V(x)$ $(\because V(b) = 0)$

$V(b) = E(b^2) - (E(b))^2 = b^2 - b^2 = 0)$

$\boxed{V(\text{constant}) = 0}$

(4) If x & y dependent r.v

- $V(x+y) = V(x) + V(y) + 2\,\text{cov}(x, y)$
- $V(x-y) = V(x) + V(y) - 2\,\text{cov}(x, y)$

$\text{cov}(x,y) = E(x \cdot y) - E(x) \cdot E(y)$

Covariance of x & y

If a & b constant

$\text{cov}(a,b) = E(a \cdot b) - E(a)E(b)$

$= ab - ab = 0$

ie Co-Variance of mutually independent Variables is zero (converse is not true)

- $V(x) \geq 0$ (always non negative)
  $E(x^2) \geq (E(x))^2$
- If $E(x^2) < (E(x))^2$ then $V(x)$ doesn't exist
- $E(x^2) = (E(x))^2$; x = constant
- $E(x^2) > (E(x))^2$; x = Variable set

Q:- Find the expectation & variance for the face values on the die?

x (rv)	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

$E(x) = \sum_{x=1}^{6} x P(x) = \frac{1}{6}\left(\sum_{x=1}^{6} x\right) = \frac{1}{6} \times \frac{n(n+1)}{2}$

$= \frac{1}{6} \times \frac{6 \times 7}{2} = \frac{7}{2} = 3.5$

face may be 4 or 3

$V(x) = E(x^2) - (E(x))^2$

$= \sum_{x=1}^{6} x^2 P(x) - (3.5)^2$

$= P(x) \sum_{x=1}^{6} x^2 - (3.5)^2$

$= \frac{1}{6} \times \frac{n(n+1)(2n+1)}{6} - (3.5)^2$

$= \frac{1}{6} \times \frac{6 \times 7 \times 13}{6} - (3.5)^2 = \frac{35}{12} \simeq 3$

Variance $\geq$ difference of any rv with in the data from the mean value

ie here difference btw^n each rv & $E(x) = 3.5$ is always less than 3

Note:-

The expectation & variance for ~~some~~ sum of the number on the dice are

$E(x) = \frac{7n}{2}$

$V(x) = \frac{35}{12} n$

n = no. of dice

Eg:- 6 dice are rolled. Find the expectation & variance for ~~some~~ sum of the number on them

$E(x) = \frac{7 \times 6}{2} = 21$

$V(x) = \frac{35 \times 6}{12} = \frac{35}{2}$

Q:- If 'x' is a discrete random variable & its cumulative probability funcⁿ is
(Probability ↑ step by step)

X:	1	2	3	4	5
P(x):	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$

$(\Sigma P(x) \geq 1)$ Cumulative.

(we need individual Probability)

find (i) $E(x)$ & $V(x)$

(ii) $E(2x+5)$, $V(-\frac{x}{2}+\frac{1}{2})$

(iii) $P(X > 2.6 / X \leq 4.7)$

An:-

X	1	2	3	4	5	
P(x)	1/5	1/5	1/5	1/5	1/5	

uniform distbⁿ

If DRV with uniform distbⁿ

$$E(x) = \frac{n+1}{2}$$

$$V(x) = \frac{n^2-1}{12}$$

(i) $E(x) = \frac{5+1}{2} = \underline{\underline{3}}$

$V(x) = \frac{25-1}{12} = \underline{\underline{2}}$

(ii) $E(2x+5) = 2E(x)+5 = 6+5 = \underline{\underline{11}}$

$V(-\frac{x}{2}+\frac{1}{2}) = \frac{1}{4}V(x) + V(\frac{1}{2})$

$= \frac{1}{4} \times 2 = \underline{\underline{\frac{1}{2}}}$

(iii) $P(x > 2.6 / X \leq 4.7)$

$= \frac{P(x > 2.6 \cap x \leq 4.7)}{P(x \leq 4.7)}$

$= \frac{P(2.6 < x \leq 4.7)}{P(x \leq 4.7)} = \frac{P(X=3)+P(X=4)}{\sum_{x=1}^{4} P(x)}$

$= \frac{1/5 + 1/5}{4/5} = \frac{2}{4} = \underline{\underline{\frac{1}{2}}}$

Q:- A bag has 4 non defective bolts & 2 defective bolts. 4 bolts are drawn at random. Find the expectation for no. of non defective bolts.

ND 4, D 2 ⇒ 6

x no. of ND defective	4	3	2
P(x)	1/15	8/15	6/15

$\frac{4C_4 \times 2C_0}{6C_4}$, $\frac{4C_3 \times 2C_1}{6C_4}$, $\frac{4C_2 \times 2C_2}{6C_4}$

$\Sigma P(x) = 1$ So $E(x)$ exists.

E(no. of nondefective bolts)

$= 4 \times \frac{1}{15} + 3 \times 8/15 + 2 \times \frac{6}{15} = \frac{40}{15} = \underline{\underline{\frac{8}{3}}}$

Q:- A player tosses 3 coins. He wins 8 rps if 3 heads occur, 3 rps if 2H occur, 1 rps for 1H occur. How much he looses if no head occur when the game is fair.

An:- rv = no. of heads (x)

x	3 (+8)	2 (+3)	1 (+1)	0 (−x)
P(x)	1/8 (3C3)	3/8 (3C2)	3/8 3C1	1/8 3C0

Value of game = Gain − loss

fair / balanced game means value of game = 0

ie gain = loss

gain − loss = 0

$(8 \times \frac{1}{8}) + (\frac{3}{8} \times 3) + (\frac{3}{8} \times 1) - x \times \frac{1}{8} = 0$

$\frac{1}{8}(8+9+3-x)=0$

$\underline{x=20}$

Q:- If x is DRV & its probability funcn is $P(x)=\frac{A}{2^x}$; $x=1,2,3\cdots\infty$

(i) find the value of A

(ii) E(X)

An:- $\sum_{1}^{\infty} P(x)=1$

$A\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^\infty}\right)=1$

$\frac{A}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^\infty}\right)=1$

$\frac{A}{2}\left(\frac{1}{1-1/2}\right)=1$

$\underline{A=1}$

(ii) $E(x)=\sum_{x=1}^{\infty} x\frac{1}{2^x} = 1\times\frac{1}{2}+2\times\frac{1}{2^2}+3\times\frac{1}{2^3}+\cdots$

$=\frac{1}{2}\left(1+2\times\frac{1}{2}+3\times\frac{1}{2^2}+\cdots\right)$

$=\frac{1}{2}\times\frac{1}{(1-\frac{1}{2})^2} = \frac{1}{2}\times\frac{1}{(1/4)}=\underline{2}$

Q:- If X is a CRV & its probability density funcn is

$f(x)=\frac{k}{x^u}$; $u>1$, $x\geq 1$ find k?

An:- $\int_{-\infty}^{\infty} f(x)\,dx=1$

$\int_{1}^{\infty}\frac{k}{x^u}dx=1$

$k\left[\frac{x^{-u+1}}{-u+1}\right]_1^\infty = k\left(0-\frac{1}{-u+1}\right)$

$\Rightarrow k\times\frac{1}{u-1}=1$

$\underline{k=u-1}$ (always +ve)

Q:- If 'x' is a CRV & its probability density funcn is

$f(x)=k|x|$ where $|x|<1/2$

(i) find k (ii) E(x) & V(x)

(iii) $E((x+2)^2)$

(iv) $S.D(\sqrt{8}x-\frac{1}{\sqrt{6}})$

An:- $\int_{-\infty}^{\infty} f(x)\,dx=1$

$\boxed{\begin{array}{c}|x|<\frac{1}{2}\\ -\frac{1}{2}<x<\frac{1}{2}\end{array}}$

$=\int_{-1/2}^{1/2} k|x|\,dx$

$=2\int_{0}^{1/2} kx\,dx$

$=2k\times\left(\frac{x^2}{2}\right)_0^{1/2} = 2k\times\frac{1}{8}=1$

$\underline{k=\frac{8}{2}=4}$

$\int_{-a}^{a} f(x)\,dx=0$ → odd

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ → Even

- O·E = O
- O·O = E
- E·E = E

- Even funcn - symmetric to y-axis (mirror image)
- odd funcn symmetric to x-axis (water image)

ii) $E(x)=4\int_{-1/2}^{1/2} x|x|\,dx$ → odd funcn

$=4\times 0 = \underline{0}$

$E(x^2)=4\int_{-1/2}^{1/2} x^2|x|\,dx$ → even funcn

$=8\int_{0}^{1/2} x^3\,dx = 8\times\left(\frac{x^4}{4}\right)_0^{1/2} = 2\times\frac{1}{2^4}$

$=\underline{1/8}$

$V(x)=E(x^2)-(E(x))^2=\frac{1}{8}-0=\underline{1/8}$

(iii) $E((x+2)^2) = E(x^2+4x+4)$

$= E(x^2)+4E(x)+E(4)$

$= \frac{1}{8} * (4\times 0) + 4 = \frac{33}{8}$

(iv) $SD(\sqrt{8}x - \frac{1}{\sqrt{6}}) = \sqrt{V(\sqrt{8}x - \frac{1}{\sqrt{6}})}$

$= \sqrt{8V(x) - 0} = \sqrt{8\times\frac{1}{8}} = 1$

Q:- If 'x' is a crv & its Probability density funcⁿ is $f(x) = \frac{k}{x^2}$; $x \geq 50$

(i) K?

(ii) $P(X \geq 100.5)$

(iii) $P(75.5 < x < 85.5)$

(iv) $P(X \leq 90.5)$

An:- $\int_{-\infty}^{\infty} f(x)dx = 1$

(i) $\int_{50}^{\infty} \frac{k}{x^2} dx = 1$

$k \times \left[-\frac{1}{x}\right]_{50}^{\infty} = 1 \Rightarrow k\left(-0+\frac{1}{50}\right) = 1$

$K = 50$

(ii) $P(x \geq 100.5) = P(X > 100.5) = \int_{100.5}^{\infty} \frac{50}{x^2} dx$

$= 50 \times \left(-\frac{1}{x}\right)_{100.5}^{\infty} = 50 \times \frac{1}{100.5} = \frac{50}{100.5}$

$= 0.497$

(iii) $P(75.5 < x < 85.5)$

$= \int_{75.5}^{85.5} \frac{50}{x^2} dx = 50\left(-\frac{1}{85.5} + \frac{1}{75.5}\right)$

$= 0.077$

(iv) $P(x \leq 90.5) = P(x < 90.5)$

$\int_{50}^{90.5} \frac{50}{x^2} dx = 50\left(-\frac{1}{90.5} + \frac{1}{50}\right) = \frac{40.5}{90.5}$

$= 0.447$

Q:- If 'x' is the r.v & probability density funcⁿ $f(x) = ax - b$; $0 < x < 1$

$E(x) = \frac{1}{2}$ find a, b?

An.- $\int_0^1 (ax-b)dx = 1$

$\left[\frac{ax^2}{2} - bx\right]_0^1 = 1 \Rightarrow \frac{a}{2} - bx = 1$

$a - 2b = 2 \rightarrow (1)$

$E(x) = \int_0^1 x(ax-b)dx = \frac{1}{2}$

$\left[\frac{ax^3}{3} - \frac{bx^2}{2}\right]_0^1 = 1/2$

$\frac{a}{3} - \frac{b}{2} = \frac{1}{2} \Rightarrow 2a - 3b = 3 \rightarrow (2)$

$2a - 4b = 4$

$b = 1$

$a = 0$

Binomial distribution

If 'x' is a binomial random variable it allows the values from '0' to 'n' with the parameters n, p. Then the probability mass funcⁿ is

$$B(x; n, p) = P(x) = \begin{cases} nC_x p^x q^{n-x} ; 0 \le x \le n \\ p+q=1 ; q = 1-p \\ = 0 \quad \text{otherwise} \end{cases}$$

- Conditions

① observations are independent (with replacement)

② Probability of success is constant.

③ mean > variance.

Values :-

- mean $= E(X) = np$
- Variance $V(X) = npq$

Q:- Two coins are tossed 5 times. Find the P of getting a heads.

(i) Exactly 3 times.

(ii) atleast one time

(iii) atmost one time.

Ans:- $n = 5 \quad p = \frac{1}{4} \quad q = 3/4$

(i) $P(X=3) = 5C_3 \times \left(\frac{1}{4}\right)^3 \times \left(\frac{3}{4}\right)^2$

$= \frac{5!}{3! \times 2!} \times \frac{9}{4^5} = \frac{90}{4^5}$

(ii) $P(\text{atleast } 1) = P(X \ge 1) = 1 - P(X=0)$

$= 1 - \left[5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5\right] = 1 - \left(\frac{3}{4}\right)^5$

(iii) $P(X \le 1) = P(X=0) + P(X=1)$

$= 5C_0 \times \left(\frac{1}{4}\right)^0 \times \left(\frac{3}{4}\right)^5 + 5C_1 \times \frac{1}{4} \times \left(\frac{3}{4}\right)^4$

$= \left(\frac{3}{4}\right)^4 \left(\frac{3}{4} + \frac{5}{4}\right) = \left(\frac{3}{4}\right)^4 \times 2$

Q:- The probability of no. of defectives in a lot is 30%, there are 4 items. Find the 'P' that

we can use binomial distbⁿ

(i) Exactly 3 defectives

(ii) atmost 2 defectives.

Ans:- (i) $P(X=3) = 4C_3 (0.3)^3 (0.7)^1$

$= 2.8 \times (0.3)^3$

(ii) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= 4C_0 \times (0.3)^0 \times (0.7)^4 + 4C_1 \times 0.3 \times (0.7)^3 + 4C_2 (0.3)^2 (0.7)^2$

$= (0.7)^4 + 1.2 \times 0.7^3 + 6 \times 0.09 \times 0.7^2$

$= 0.7^4 + 1.2 \times 0.7^3 + 0.54 \times 0.7^2$

Q:- $E(X) = 3$; $V(X) = 2$ find

(i) $P(X \le 0.9)$

(ii) $E(X^2+6)$

(iii) $SD\left(\frac{x}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$

[mean > variance. So binomial distbⁿ can be used]

Ans:- $np = 3$

$npq = 2$

$q = 2/3$

$p = 1 - \frac{2}{3} = \frac{1}{3}$

$n = \frac{3}{(1/3)} = 9$

(i) $P(X\le 0.9) = P(X<0.9) = P(X=0)$

$$= {}^{9}C_0 \times \left(\frac{1}{3}\right)^0 \times \left(\frac{2}{3}\right)^9 = \left(\frac{2}{3}\right)^9$$

(ii) $E(X^2+6) = E(X^2)+6$

$E(X^2) = ?$

$V(X) = E(X^2) - (E(X))^2$

$E(X^2) = V(X) + (E(X))^2 = 2+9 = 11$

$E(X^2+6) = 11+6 = 17$

(iii) $SD\left(\frac{X}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \sqrt{V\left(\frac{X}{\sqrt{2}}\right) - V\left(\frac{1}{2}\right)}$

$$= \sqrt{\frac{1}{2}V(x)} = \sqrt{\frac{1}{2}\times 2} = 1$$

Q:- 'X' is a binomial random variable then find the value of $\sum_{x=0}^{n} \frac{x}{n}\, {}^{n}C_x p^x q^{n-x}$

Ans:- $\frac{1}{n}\sum_{x=0}^{n} x \cdot (P_x)$

$$= \frac{1}{n}E(x) = \frac{1}{n}\times np = p$$

○ If X and Y are independent with parameter (n, p) & (m, p) respectively. Then $X+Y$ is a binomial random variable with parameters. $(n+m, p)$

○ If X & Y are independent poisson random variables with parameters λ_1 & λ_2 then $X+Y$ is a poisson random variable with parameter $\lambda_1 + \lambda_2$.

Poisson Distribution

(Arrival Rate, defect item, Time dependence, Rare occurence)

$$p(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & \lambda > 0,\ 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Definition:- If x is a poisson random variable defined in the interval from 0 to ∞ with the parameter λ $(\lambda > 0)$ then the probability mass function is $p(x;\lambda>0) = p_x$

Conditions:-

→ Observations are independent and very large $(n \to \infty)$

→ either probability is success is very small $(p \to 0)$

→ $np = \lambda \Rightarrow p = \lambda/n$

Values

Mean $= E(X) = \lambda$

Variance $= V(X) = \lambda$

Note:- Mean = Variance = parameter = λ

→ Sum of the independent poisson random variable is also a poisson random variable

Q) On an avg 30 calls are received from 8 AM to 10 AM. Find the probablity for a period of 30 minutes

i) No call is received
ii) Exactly 3 calls are received
iii) Atmost 2 calls are received.

Arrival rate
Time Dept

Ans) Total 120 minutes — 30 calls

$P(\text{call in a min}) = \frac{30}{120}$

$n = 30\,\text{min} \Rightarrow np = \lambda = \frac{30^2}{120} = 7.5$

Otherwise

$30 =$ Avg call for 120 min

$30 =$ Expectance for 120 min

$\lambda = 30$ for 120 min

$\lambda = \frac{30}{120}$ for 1 min

$\lambda = \frac{30^2}{120}$ for 30 min

i) $P(X=0) = \frac{e^{-7.5}\cdot(7.5)^0}{0!} = e^{-7.5}$

ii) $P(X=3) = \frac{e^{-7.5}(7.5)^3}{3!} = 0.03888$

iii) $P(X\le 2) = P(X=0) + P(X=1) + P(X=2)$

$= e^{-7.5} + e^{-7.5}(7.5) + \frac{e^{-7.5}\times 7.5^2}{2}$

$P(X\le 2) =$

Q) The probablity of defect item is 4×10^{-5}. There are 10000 items, find the probablity that 1) None of items are defective 2) Atleast 2 defects 3) Atmost 1 defect.

N = large
defect prob. very small.

$n = 10000$

$p = 4\times10^{-5}$

$\lambda = 0.4$

Ans) 1) $P(X=0) = \frac{e^{-0.4}\times 0.4^0}{0!} = e^{-0.4}$

2) $P(X\ge2) = 1 - P(X<2) = 1 - [P(0)+P(1)]$

$P(X\ge2) = 1 - (e^{-0.4} + e^{-0.4}\times0.4) = 1 - 1.4e^{-0.4}$

3) $P(X=0, X=1) = 1.4e^{-0.4}$

Q) If X is a poisson random variable, given that $P(X=2) = P(X=3)$

Find i $P(X<1.3)$ ii) $E(X^2+3)$ iii) $SD\left(\sqrt{3}x - \frac{1}{\sqrt{7}}\right)$

Ans) $\frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}\lambda^3}{3!} \Rightarrow \boxed{\lambda = \frac{3!}{2!} = 3}$

mean $= 3$

Variance $= 3$

$E(X^2) = V(X) + E(X)^2$

$E(X^2) = 3 + 9 = 12$

ii) $E(x^2+3) = E(x^2)+3 = 12+3 = \underline{\underline{15}}$

i) $P(X\le 1.3) = P(0)+P(1) = e^{-3}+e^{-3}\cdot 3 = \underline{\underline{4e^{-3}}}$

iii) $SD(\sqrt{3}x - 1/\sqrt{7}) = \sqrt{V(\sqrt{3}x - 1/\sqrt{7})} = \sqrt{V(\sqrt{3}x)} = \sqrt{3\cdot V(x)} = \sqrt{9}$

$SD\left(\sqrt{3}x - \frac{1}{\sqrt{7}}\right) = \underline{\underline{3}}$

Q) If X is a poisson random variable & $E(x^2)=12$ then, Find $V(2x-3)$ $E((x+1)^2)$

i) $V(2x-3) = 4V(x) = \cancel{4\times 12 = 48} = 4\times 3 = \underline{\underline{12}}$

ii) $E((x+1)^2) = E(x^2+2x+1)$

$= E(x^2)+2E(x)+1$

$= 12+6+1$

$E((x+1)^2) = \underline{\underline{19}}$

given $E(x^2)=12$

$\lambda = E(x^2)-\lambda^2$

$\lambda^2+\lambda-12=0$

$\lambda = \frac{-1\pm\sqrt{1+48}}{2}$

$\lambda = \frac{-1\pm 7}{2} = -4, 3$

-4 Ruled out $\lambda > 0$

Normal Distribution [Gaussian distribution]

Definition :- If X is a normal random variable defined in interval $-\infty$ to $+\infty$ with mean μ and variance σ^2 then the probablity density function is

$$N(x:\mu,\sigma^2) = f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < x < +\infty \\ & -\infty < \mu < \infty \\ & 0 < \sigma < \infty \\ 0 & \text{otherwise} \end{cases}$$

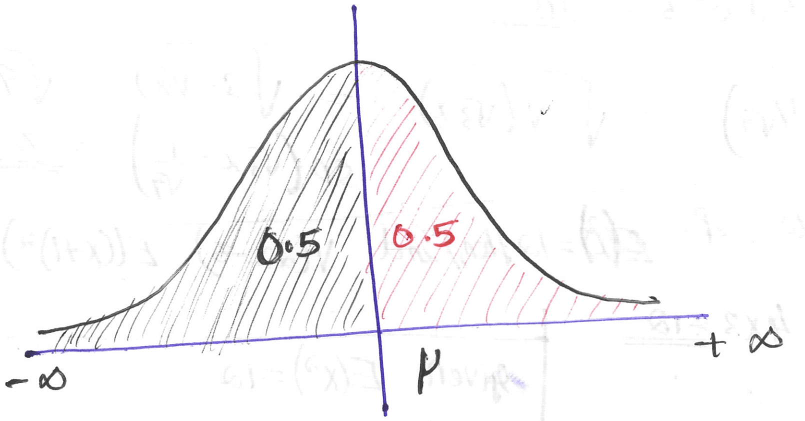
STANDARD NORMAL RANDOM VARIABLE

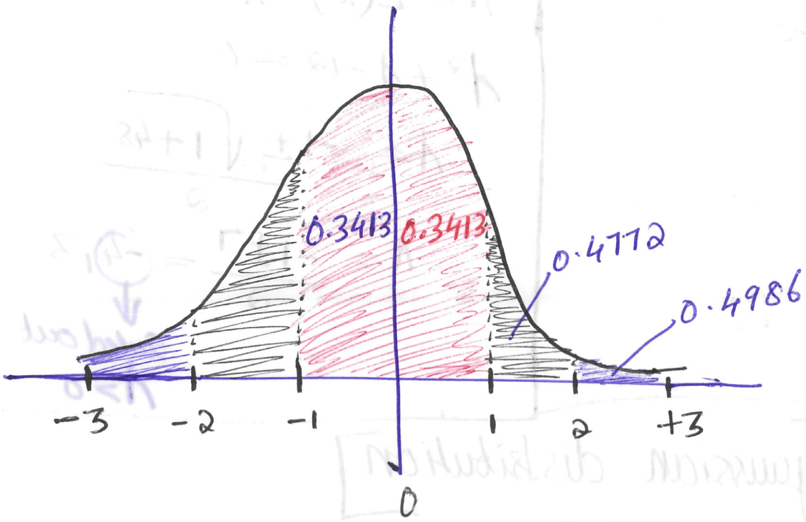
$$N(z,0,1) = f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Mathematically $z = \frac{X-E(X)}{\sqrt{V(X)}} = \frac{X-\mu_x}{\sigma_x}$

$\boxed{-3 < z < 3}$ range with sharpness.

If z is a normal variable with mean 0 & variance 1, then the random variable is known as Standard normal random variable. Its density function is $N(z,0,1)$





$P(a<x<c)$

$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)} dz$$

put $z = \frac{x-\mu}{\sigma}$

$\sigma z = x-\mu$

$\sigma\, dz = dx$

$z_a = \frac{a-\mu}{\sigma}$

$z_b = \frac{b-\mu}{\sigma}$

$$\Rightarrow p(a<x<c) = \int_{z_a}^{z_b} \frac{\cancel{\sigma}}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$P(a<x<b) = p(z_a<z<z_b)$

$N(x;\mu,\sigma^2) \Longleftrightarrow N(z,0,1)$

Q) X is normally Distributed with $\mu = 50$ and $\sigma = 10$

Find
i) $P(X<60)$
ii) $P(X>40)$
iii) $P(|X-50|<10)$
iv) $P(X<50)$

Ans

i) $P(X<60) = P\left(\frac{X-\mu_X}{\sigma_X} < \frac{60-50}{10}\right)$

$P(X<60) = P(Z<1) = 0.5 + 0.3413$
$= \underline{0.8413}$

ii) $P(X>40) = P(Z>-1)$
$= \underline{0.8413}$

iii) $P(|X-50|<10)$
$= P(|Z|<1)$
$P(-1<Z<1) = 2 \times 0.3413$
$P(|X-50|<10) = \underline{0.6826}$

iv) $P(X<50)$
$= P(Z<0) = \underline{0.5}$

If X & Y are independent random variables that are normally distributed with parameters (μ_1, σ_1^2) & (μ_2, σ_2^2) then X+Y is normally distributed with parameters $(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

(mean add
Variance add)

Variance $= \sigma^2$
SD $= \sigma$

Q) X is normally distributed with $\mu = 20$ and $S.D = 3.33$ find the probability b/w 21.11 & 26.66

Ans) $P(21.11 < X < 26.66) = P(1/3 < Z < 2) = 0.4772 - P(1/3)$

$= 0.4772 - 0.1293$

$P\left(\frac{1}{3} < Z < 2\right) \quad = \underline{0.3479}$

- Normal Random Variable is symmetrical about mean and shape of the curve is bell shaped curve.
- Normal probability at a particular point is zero.
- Sum & differences b/w independent normal random variable is also a normal random variable.
- Binomial distribution is approximated to normal if observations are infinitely large, neither probability of success nor probability of failure are very small.

Correlation & Regression

- The degree of relation b/w the variables is known as correlation
- The changes in one variable are affecting the changes in other variable parallelly then those variables are known as correlated variables.
- If the changes in both variables are in same direction, then those variables are known as +vely correlated variables.
- If the changes in one variable are affecting the changes in other variable in reverse direction then those variables are known as negatively correlated variables.
- Mathematically correlation is defined as

$$\boxed{r(x,y) = \frac{cov(x,y)}{\sigma_x \sigma_y}}$$

r = correlation coefficient

$\boxed{r \neq 0}$

$$\boxed{-1 \leq r(x,y) \leq 1}$$

- If X & Y are independent random variables then $cov(x,y) = 0$

$\Rightarrow r(x,y) = 0 \Rightarrow$ Highly uncorrelated.

- But converse is not true ie) if $r = 0 \not\Rightarrow$ independent

Regression

The functional relationship b/w the correlated variables is known as regression.

Regression lines

Y on X

$$(y-\bar{y}) = \boxed{r \cdot \frac{\sigma_y}{\sigma_x}}(x-\bar{x})$$

→ b_{yx} regression coefficient.

X on Y

$$(x-\bar{x}) = \boxed{r \cdot \frac{\sigma_x}{\sigma_y}}(y-\bar{y})$$

→ b_{xy} regression coefficient

Properties

1) $b_{yx} \cdot b_{xy} = r^2$

$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

2) $|b_{yx}|>1 \Rightarrow |b_{xy}|<1$ since $|r| \leq 1$

3) $b_{yx} = b_{xy} \Rightarrow \sigma_y = \sigma_x$

Note:

→ Both regression coefficients must have same sign.

→ Both +ve ⇒ r +ve ; Both -ve ⇒ r -ve

→ If they are opposite sign then r doesnot exist.

→ Regression equations passes through the point $\bar{X}, \bar{Y}$

Q) The regression eq are $3x+4=1$, $x+34=0$ find value of r, find $\bar{x}$ & $\bar{y}$.

$3x = 1-y$ | $x+3y=0$
$-3x = y-1$ | $3y = -x$
$\Rightarrow b_{yx} = -3$ | $-3y = x$
$\bar{y} = 1$

highest coefficient = dependent variable

x on y | y on x

$3x+y=1$ | $y = \frac{-1}{3}(x)$ | $\Rightarrow r = -\sqrt{\frac{1}{3} \cdot \frac{1}{3}} = -\frac{1}{3}$

$x = \frac{1}{3} - \frac{y}{3}$ | $b_{yx} = -1/3$

$x = \frac{-1}{3}(-1+y)$ | $\bar{x} = 0$

$b_{xy} = \frac{-1}{3}$ | $\bar{y} = 0$

$\bar{y} = \cancel{1}$

$\bar{x} = 0$

$3x+y \quad 1$
$3x+9y \quad 0$
$8y = -1$
$y = -1/8$

$\bar{x}, \bar{y}$ points satisfy both regression equations.
they are points of intersection of regression equations.

$$\boxed{\int_0^\infty x^n e^{-x}\, dx = n!}$$

UNIFORM RECTANGOLAR DISTRIBUTION

$$\boxed{f(x) = \begin{cases} \frac{1}{(b-a)} & a<x<b \\ 0 & \text{otherwise} \end{cases}}$$

$\boxed{\text{mean } \mu = \frac{b+a}{2}}$ $\boxed{\text{Variance } V(x) = \sigma^2 = \frac{(b-a)^2}{12}}$

Exponential Distribution

$f(x) = \theta e^{-\theta x} \quad x \geq 0 \quad \theta > 0$
$= 0$ otherwise

for random variable distribution from 0 to ∞. Continous!

$\boxed{\text{mean } E(x) = \mu = 1/\theta}$

$\boxed{\text{Varian } V(x) = \sigma^2 = 1/\theta^2}$

Direct Corelation +ve
Indirect Corelation -ve
Perfect corelation ± 1

$$r = \frac{\text{cov}\,x,y}{\sigma_x \sigma_y}$$

$$\boxed{\text{cov}(x,y) = \frac{1}{n}\sum[xy - \mu_x\mu_y]}$$

$$\boxed{r = \frac{\sum(x-\mu_x)(y-\mu_x)}{\sqrt{\sum(x-\mu_x)^2(y-\mu_y)^2}}}$$

If $r=0$ the two lines of regression are perpendicular
$x=\bar{x}$ $y=\bar{y}$

if $r=\pm 1$ two lines coincide.

Angle between lines of regression

$$\boxed{\theta \Rightarrow \tan\theta = \frac{1-r^2}{|r|} \cdot \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}}$$

$r=0$ $\tan\theta = \infty \Rightarrow \theta = \pi/2$

$r=\pm 1$ $\tan\theta = 0,\pi$ $\underline{\theta = 0,\pi}$

$$\boxed{\text{Coefficient of Variance} = \frac{\sigma}{\mu}}$$

Example misc question
Ex Q 53, 49, 38, 23, 19
in text book

Statics
Mean = avg
Mode = highest repeated
Meaden = Ascending order
middle or
mean of middle

SD = σ

SD = σ Probabiliti
mean = expectance
median = split into two equal area
$\boxed{P(x<a) = P(x>a) = 1/2}$
median
Mode = Highest probablity
Value of X for Highest f(x)

$$\tan\theta = \frac{1-r^2}{|r|}\,\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2} = \frac{1-r^2}{|r|}\,\frac{\frac{\sigma_x}{\sigma_y}}{\left(\frac{\sigma_x}{\sigma_y}\right)^2+1} = \frac{1-r^2}{|r|}\,\frac{\frac{b_{xy}}{r}}{\left(\frac{b_{xy}}{r}\right)^2+1}$$

$$r = \sqrt{b_{xy}b_{yx}} \qquad \left(\frac{\sigma_x}{\sigma_y}\right) = \frac{b_{xy}}{r}$$

$$\boxed{\tan\theta = \frac{1-r^2}{|r|}\,\frac{r\,b_{xy}}{(b_{xy})^2+r^2}}$$

$\Longrightarrow E(x^2+y^2)$ is asked $\rightarrow$ $E(x^2) + E(y^2) = \underline{E(x)^2 + V(x) + E(y)^2 + V(y)}$

eg: Q page 190-42 Test book

$\rightarrow$ In questions with binomial or poison identify the random variable
$\rightarrow$ identify the max value of X ie n
$\rightarrow$ identify p. If X = no. of (particular event happenig)
the p is the probablity that even happens
$\rightarrow$ If expectance is given take as np or d.
$\rightarrow$ If X is unbounded $\rightarrow$ take poison.